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Automation, Growth, and Factor Shares in the Era of Population Aging

Abstract

How does population aging affect economic growth and factor shares in times of increasingly automatable production processes? The present paper addresses this question in a new macroeconomic model of automation where competitive firms perform tasks to produce output. Tasks require labor and machines as inputs. New machines embody superior technological knowledge and substitute for labor in the performance of tasks. Automation is labor-augmenting in the reduced-form aggregate production function. If wages increase then the incentive to automate becomes stronger. Moreover, the labor share declines even though the aggregate production function is Cobb-Douglas. Population aging due to a higher longevity reduces automation in the short and promotes it in the long run. It boosts the growth rate of absolute and per-capita GDP in the short and the long run, lifts the labor share in the short and reduces it in the long run. Population aging due to a decline in fertility increases automation, reduces the growth rate of GDP, and lowers the labor share in the short and the long run. In the short run, it may or may not increase the growth rate of per-capita GDP, in the long run it unequivocally accelerates per-capita GDP growth.

JEL-Codes: E220, J110, J220, J230, O330, O410.

Keywords: population aging, automation, factor shares, endogenous technical change, endogenous labor supply.

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1 Introduction

Automation, i. e., the use of machines to replace and complement human beings in the performance of tasks, has been a key driver of economic growth since the beginning of the industrial revolution (Landes (1969), Mokyr (1990), Allen (2009)). At least since the early 1960ies, this tendency has become more pronounced as technological advances in areas like robotics, information technology, digital technology, and artificial intelligence substantially widened the scale and scope of automation (Brynjolfsson and McAfee (2014), Ford (2015), Ross (2016), Goldfarb and Tucker (2019)).

For many industrialized countries the period from 1960 to today has also been an era of *population aging* that is predicted to extend further into the 21st century (Lutz, Sanderson, and Scherbov (2008), United Nations (2015)). Since longevity has increased and fertility fallen, older individuals have become a proportionally larger fraction of the total population (Weil (2008)).[2]

The focus of this paper is on the effect of population aging on automation, growth, and factor shares. Unlike existing studies, my analysis shows that the source of aging and the considered time horizon matter for the direction of impact. On the one hand, an increase in longevity reduces automation in the short and promotes it in the long run. It boosts the growth rate of absolute and per-capita GDP in the short and the long run, lifts the labor share in the short and reduces it in the long run. On the other hand, a decline in fertility increases automation, reduces the growth rate of GDP, and lowers the labor share in the short and the long run. While in the short run, it may or may not increase the growth rate of per-capita GDP, it unequivocally accelerates per-capita GDP growth in the long run.

I derive these findings in a novel competitive one-sector endogenous growth model. The design of the production sector is the central conceptual innovation of this paper. Here, I distinguish fixed capital from machines. Automation refers to the substitution of new and better machines for hours worked by labor in the performance of tasks. A higher (expected) wage strengthens the incentive to automate.[3] While automation is labor-saving at the level of individual tasks, it is labor-augmenting in the economy's reduced-form

[2]Section B.1 of the Online Appendix provides empirical evidence on the increase in longevity and the decline in fertility since the 1960ies for a sample of 27 selected OECD countries. The evolutions shown there in Figures B.1 and B.2 also support key assumptions regarding the household sector introduced below.

[3]This mechanism mimics a key finding of the so-called induced innovations literature of the 1960s: higher expected wages induce faster labor-saving technical change (see, Hicks (1932), von Weizsäcker (1962), Kennedy (1964), Samuelson (1965), Drandakis and Phelps (1966), or Funk (2002)). It also plays an important role in models where automation is not labor-augmenting (see, e. g., Acemoglu and Restrepo (2018d) or Zeira (1998)). The production sector developed in the present paper builds on and extends the one devised in Irmen (2017) and Irmen and Tabaković (2017) where the focus is on endogenous capital- and labor-saving technical change. Irmen (2020) studies the link between these models and the taxonomy developed in Acemoglu (2010).

aggregate production function. Even though the latter is Cobb-Douglas with fixed parameters, automation affects the labor share.

The household sector features two-period lived overlapping generations. Individuals face a survival probability when they enter the second period of their lives. Population aging corresponds to an increase in this probability and/or to a decline in fertility. The per-period utility function of individuals is of the generalized log-log type recently proposed by Boppart and Krusell (2020). Hence, the individual supply of hours worked is endogenous, falls in the real wage, and declines at a constant rate in response to a constant wage growth (Irmen (2018b)). To the best of my knowledge, the present paper is the first that studies Boppart-Krusell preferences in a fully-fledged endogenous growth model. This is the second conceptual innovation of this paper.

The economy converges to a steady-state path that is consistent with Kaldor's stylized facts (Kaldor (1961)). New machines embody improved technological knowledge that accumulates through periodic automation investments. This is the source of sustained growth. A steady state is feasible since technological knowledge is labor-augmenting in the economy's reduced-form net production function (Irmen (2018a)). In addition, and in line with recent empirical evidence, the amount of hours worked per worker declines at a constant rate (Huberman and Minns (2007), Boppart and Krusell (2020)).

As a first set of results, my analysis uncovers how the possibility of automation affects the behavior of firms and the labor share. At the level of the individual task automation gives rise to a *rationalization effect* as fewer working hours are needed to perform a given task. At the same time, automation lowers the cost per task. I refer to the latter as the *productivity effect* of automation. It induces a *task expansion effect*, i. e., automating firms increase the set of tasks they perform. The slope of the aggregate demand for hours worked reflects how both the rationalization and the task expansion effect respond to a change in the real wage.

I find that automation reduces the labor share. Without automation the share in the value added that accrues to tasks coincides with the labor share. However, with automation investments these shares differ, and the labor share declines. Moreover, the stronger the incentives to automate the lower is the labor share. Nevertheless, even if these incentives become very strong the labor share remains bounded away from zero.

The main results of my analysis concern the direction of impact and the mechanics through which population aging affects automation, growth, and factor shares in the short and in the long run. These findings may be sketched as follows.

Young individuals who expect to live longer want to increase their consumption possibilities in old age. Therefore, they expand their supply of working hours and save a larger fraction of their earnings. In the short run, these behavioral adjustments increase the aggregate supply of hours worked at the intensive margin and lower the equilibrium wage. As a consequence, the incentive of firms to engage in automation investments falls, and the labor share rises. GDP is affected through two channels. On the one hand, the level of employment, hence, GDP increases. On the other hand, the weakened incentives to automate imply that the productivity of labor in the performance of tasks falls. However, as firms choose the degree of automation and the amount of performed tasks optimally, the latter channel has no first-order effect on GDP. Hence, GDP increases in the short run. Since the labor supply expands at the intensive margin, per-capita GDP rises, too.

The long-run effects of a permanent increase in longevity materialize through an increase in the savings rate. This stimulates the accumulation of fixed capital and allows for higher wages. As the working hour becomes more expensive, firms respond with more automation. In steady state, this speeds up the growth rate of absolute and per-capita GDP and reduces the labor share.

The short-run effects of population aging through a decline in the fertility rate materialize in the period following the decline when the labor supply shrinks at the extensive margin. This leads to a higher equilibrium wage, stronger incentives to automate, and hence, to a lower labor share. GDP is again affected through two channels. On the one hand, the level of employment, hence, GDP declines. On the other hand, the strengthened incentives to automate imply that the productivity of labor in the performance of tasks increases. However, the latter channel has again no first-order effect on GDP as firms choose the degree of automation and the amount of performed tasks optimally. Hence, GDP falls in the short run. Nevertheless, if the proportionate decline in GDP is smaller than the decline in population, per-capita GDP can increase.

If the decline in fertility is permanent then the labor supply is lower in all periods following the decline. Accordingly, in these periods the appropriately defined capital-labor ratio and wages will be higher so that the incentives to automate become more pronounced. Therefore, the long run has more automation, faster growth of per-capita GDP, and a lower labor share.

The present paper is related to several strands of the literature. First, it contributes to the recent literature on endogenous automation and economic growth (see, e. g., **?**, Acemoglu and Restrepo (2018a), Acemoglu and Restrepo (2018c), Acemoglu and Restrepo (2018d), Berg, Buffie, and Zanna (2018), or Hémous and Olsen (2021)). In contrast to these contributions, my analytical framework provides a novel and tractable "neoclassical" alternative. Automation is the consequence of investments in new machines that substitute for human labor in a widening range of tasks and, nevertheless, appears as endogenous labor-augmenting technical change in the reduced-form aggregate production function. Moreover, sustained growth is due to the accumulation of technological knowledge embodied in new machines rather than to a mechanism that mimics the one of the AK-model (de La Grandville (1989), Klump and de La Grandville (2000), Palivos and Karagiannis (2010)).

Second, the present paper complements the literature on automation, economic growth, and demographic change (see, e. g., Acemoglu and Restrepo (2018b), Cutler, Poterba, Sheiner, and Summers (1990), Irmen (2017), among others). Contrary to these contributions, the focus of my research is on the link between population aging, the individual labor supply, individual savings, and the equilibrium incentives to automate. This perspective leads, e. g., to the novel insight that the qualitative effects of population aging through a higher life expectancy on automation incentives in the short and in the long run are of opposite sign.

Allowing for an endogenous supply of hours worked is key to these findings. The representation of individual preferences with a generalized log-log utility function of the Boppart-Krusell class reveals in addition that the wage elasticity of the individual supply of hours worked matters for the link between population aging and the incentive to automate. In the short run, this elasticity affects the response of the equilibrium wage to an increase in life expectancy and/or to a decline in fertility. In the long run, it is a determinant of the steady-state growth rate of technological knowledge. Therefore, it affects the impact an increase in life expectancy and/or a decline in fertility has on the steady-state growth rates of technological knowledge, per-capita and aggregate macroeconomic variables.

Third, my research contributes to the literature that aims at explaining the global decline in the labor share (see, e. g., Karabarbounis and Neiman (2014), Piketty (2014)).[4] Here, I maintain that the decline in the labor share is also a long-run consequence of automation induced by population aging. However, in contrast to other studies my analysis predicts that the labor share remains bounded away from zero even if the incentives to automate become very strong.

Finally, let me confront two key findings of the present paper with those obtained in the canonical OLG-model.[5] In the latter model, a decline in the population growth rate increases the capital-labor ratio in the long run and induces positive level effects on per-capita variables. In contrast, the present paper shows that a higher capital-labor ratio due to a decline in population growth generates incentives for automation that can trigger a higher long-run growth rate. A permanently lower population growth rate is therefore associated with faster long-run growth of per-capita variables.[6] Moreover, in the model of this paper a lower population growth rate is associated with a decline in the labor share even though the reduced-form production function is Cobb-Douglas. This contrasts with the canonical OLG-model where the Cobb-Douglas production function precludes that a lower population growth rate can have an effect on the labor share.

[4]Task-based models of automation tend to predict that automation reduces the labor share (see, e. g., Acemoglu and Restrepo (2018d), and Acemoglu and Restrepo (2018c) for a discussion of the literature). In the present paper this feature occurs in spite of a Cobb-Douglas production function with constant coefficients.

[5]The latter has two-period lived individuals with logarithmic utility, an exogenous labor supply growing at the same constant rate as the population, and a neoclassical production function of the Cobb-Douglas type (see, e. g., Acemoglu (2009), Section 9.3).

[6]This implication also contrasts with so-called semi-endogenous growth models (Jones (1995)) where permanently lower population growth reduces the long-run growth rate of per-capita variables.

The remainder of this paper is organized as follows. Section 2 presents the model. Section 2.1 introduces the novel production sector. Here, I show that the profit-maximizing production plan associates automation with a rationalization effect, a productivity effect, a task-expansion effect, and an output expansion effect. Moreover, I show that automation is labor-augmenting in the aggregate production function and reduces the labor share. Section 2.2 introduces the household sector. Sections 3, 4, and 5 contain the main results of this paper. The focus of Sections 3 is on the inter-temporal general equilibrium, the mechanics of the labor market, and the properties of the dynamical system. Sections 4 and 5 derive the effect of population aging on automation, growth, and factor shares in the short and the long run. Section 6 concludes. All proof are relegated to Section A, the Appendix. Section B is an Online Appendix. It contains supplementary empirical findings as well as a calibration exercise.

2 The Model

The economy comprises a production, a household, and an insurance sector in an infinite sequence of periods $t = 1, 2, ..., \infty$. The production sector has competitive firms that manufacture a single good. Building on Irmen (2017), Irmen and Tabaković (2017), and Irmen (2020) the production of this good requires tasks to be performed. The manufactured good may be consumed or invested. If invested, it either serves as contemporaneous *automation investments* or as future *fixed capital*.

The household sector has overlapping generations of individuals who potentially live for two periods, youth and old age. Survival into old age is stochastic. The individual lifetime utility function features a Boppart-Krusell generalized log-log utility function (Boppart and Krusell (2020)). Hence, the labor supply is endogenous. I follow, e. g., Yaari (1965) or Blanchard (1985), and assume a perfect annuity market for insurance against survival risk.

There are four objects of exchange, the manufactured good, fixed capital, labor, and annuities. Each period has markets for these objects. Firms rent fixed capital, undertake automation investments, demand labor, and supply the manufactured good. Households demand the manufactured good for consumption and savings, supply labor, and exchange savings for annuity policies. Insurance companies sell these policies and rent the savings as fixed capital to firms that use it to produce in the next period. Without loss of generality, fixed capital fully depreciates after one period. The manufactured good serves as numéraire.

Throughout, I denote the time-invariant growth rate of some variable x_t between two adjacent periods by g_x. Moreover, I often use subscripts to write first- and second-order derivatives. For instance, the notation for the derivatives of some function $G(x, y)$ would be $G_2(x, y) \equiv \partial G(x, y)/\partial y$ or $G_{21}(x, y) \equiv \partial^2 G(x, y)/\partial y \partial x$. I also write G instead of $G(x, y)$ or $G(\cdot)$ whenever this does not cause confusion.

2.1 The Production Sector

The production sector has many small firms operating under perfect competition. Their behavior may be studied through the lens of a competitive representative firm. At all t, this firm has access to the production function

$$Y_t = \Gamma K_t^{\gamma} N_t^{1-\gamma}, \quad 0 < \gamma < 1. \tag{2.1}$$

Here, Y_t denotes the total output of the manufactured good, K_t the amount of fixed capital, and N_t the amount of performed tasks. The parameter $\Gamma > 0$ reflects cross-country differences in geography, technical and social infrastructure that affect the "transformation" of fixed capital and tasks into the manufactured good.

All units included in the stock of fixed capital, and, likewise, each of the N_t tasks are treated as homogeneous, i. e., they may provide the same marginal contribution to the output of the manufactured good. However, due to the "law of a diminishing marginal product" the marginal contribution of task n is greater than the one of task n' when $n' > n$. This mimics the neoclassical artifice that captures the heterogeneity within each input aggregate with a diminishing marginal product.

The performance of tasks requires working hours and machines. These inputs are strong substitutes with an elasticity of substitution strictly greater than unity. Machines embody technological knowledge. Automation results from investments in new machines that embody improved technological knowledge and increase the productivity of labor in the performance of tasks.

2.1.1 Tasks and Technology

Let $n \in \mathbb{R}_+$ index these tasks. At t, each task is performed once. The production function of task n is

$$1 = a_t(n) h_t(n), \tag{2.2}$$

where $h_t(n)$ is working hours, and $a_t(n)$ is the productivity per hour worked on the performance of task n.[7] The latter is given by

$$a_t(n) = A_{t-1} \left(1 + q_t(n)\right), \quad q_t(n) \geq 0. \tag{2.3}$$

Here, $A_{t-1} > 0$ is an aggregate indicator of the level of technological knowledge at $t-1$ to which the firm has free access at t. To fix ideas, one may think of A_{t-1} as representing the

[7] Allowing for each task n to be performed at a scale $x_t(n) \neq 1$, means that (2.2) becomes $x_t(n) = a_t(n)h_t(n)x_t(n)$. This generalization leaves the results derived below unchanged if the corresponding investment outlays of equation (2.4) are replaced by $i_t(n)x_t(n) = \alpha q_t(n)x_t(n)$ reflecting the idea that a machine with a higher capacity requires proportionately larger investment outlays.

level of technological knowledge embodied in the last vintage of installed machines that may still be activated. The variable $q_t(n)$ is the growth rate of productivity per working hour in task n at t. A growth rate $q_t(n) > 0$ requires an automation investment in a new machine at t. This machine partially replaces labor in the performance of task n. The degree to which this substitution occurs is endogenous.[8]

The invention, construction, installation, and running of a new machine for task n gives rise to investment outlays of

$$i_t(n) = \alpha q_t(n), \quad \alpha > 0, \tag{2.4}$$

units of the contemporaneous manufactured good.[9] The parameter α parameterizes the efficiency of the activities that eventually bring the new machine into use. Investment outlays increase in the growth rate of productivity, $q_t(n)$, i. e., a machine that embodies a better technology is more expensive.

The technology described by equations (2.2) - (2.4) incorporates the notion of automation as the substitution of working hours per task, $h_t(n)$, with technological knowledge represented by $a_t(n)$. In $(a_t(n), h_t(n))$ - space equation (2.2) has an interpretation as a unit isoquant. It states the set of necessary input combinations of working hours and technological knowledge as

$$h_t(n) = \frac{1}{a_t(n)}, \quad a_t(n) \geq A_{t-1}. \tag{2.5}$$

This is illustrated in Figure 2.1. At $t = 1$, the relevant isoquant is the blue curve $h_1(n)$ starting at point $(A_0, 1/A_0)$. Since $A_0 > 0$ is given, task n requires at most $1/A_0$ working hours. The use of more technological knowledge shifts $a_1(n)$ further to the right of A_0. Accordingly, the amount of working hours shrinks along the isoquant. At $t = 2$, the relevant isoquant, $h_2(n)$, starts at $(A_1, 1/A_1)$. As depicted, $A_1 > A_0$ so that $h_2(n)$ begins to the right of A_1. Again, if more technological knowledge than A_1 is used then $a_2(n)$ moves further to the right of A_1, and the amount of working hours shrinks.

Since technological knowledge is embodied in machines the substitution of working hours per task with technological knowledge has to occur through a substitution of working hours with machines. Using $q_t(n) = i_t(n)/\alpha \geq 0$ from (2.4) in (2.5) delivers the unit

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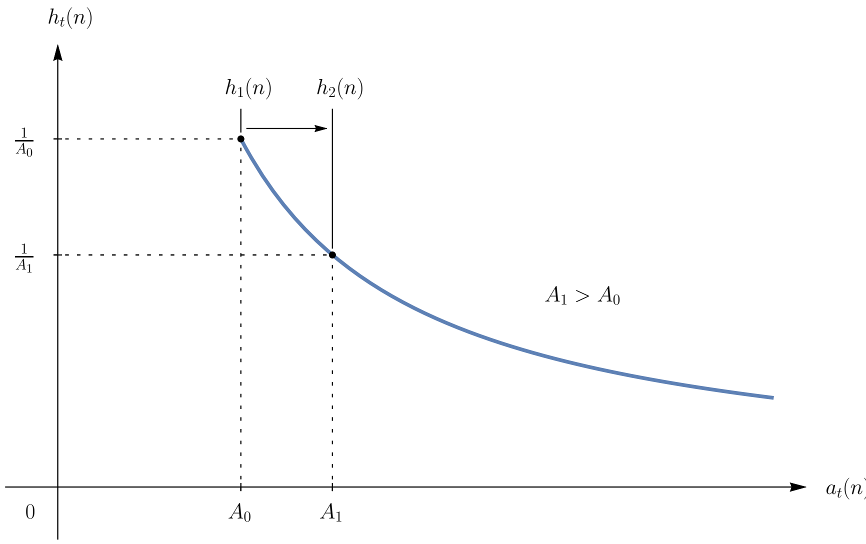
[8]Imperfect substitution of labor with machines suggests that task n comprises subtasks. Following the substitution, more of these subtasks are performed by machines. This kind of substitution is in line with recent evidence, e. g., on the effect of computer-based technologies or of machine learning on occupations (Autor, Levy, and Murnane (2003), Brynjolfsson, Mitchell, and Rock (2018)).

[9]If task n was performed in $t-1$ then $i_t(n)$ would also include the scrap costs of the old machine that is replaced. To avoid the asymmetry this would introduce for the investment outlays of tasks $n \in [0, N_{t-1}]$ and $n \in (N_{t-1}, N_t]$ if $N_t > N_{t-1}$ we neglect such expenses. This comes down to assuming that the firm can get rid of old machines without incurring a cost, i. e., its production set satisfies the property of *free disposal*. Observe that the qualitative results of this paper extend to more general functions i as long as these are increasing and convex.

Figure 2.1: Automation as the Substitution of Working Hours per Task with Technological Knowledge.



Note: The blue curve starting at $(A_0, 1/A_0)$ is $h_1(n)$. Hence, at $t = 1$ automation, i. e., the substitution of working hours per task with technological knowledge, occurs along this curve to the right of A_0. At $t = 1$, the blue curve starting at $(A_1, 1/A_1)$ is $h_2(n)$. Automation occurs along this curve to the right of A_1.

isoquant describing the set of necessary input combinations of working hours and investment outlays as

$$h_t(n) = \frac{1}{A_{t-1}\left(1 + \frac{i_t(n)}{\alpha}\right)}, \quad i_t(n) \geq 0. \tag{2.6}$$

Hence, automation investments substitute for working hours. Finally, observe that an automation investment is not an essential input in the performance of task n at t. Without such an investment task n may be performed with a machine of the past vintage that embodies the technological knowledge represented by A_{t-1}, hence $h_t(n) = 1/A_{t-1}$ if $i_t(n) = 0$.

2.1.2 Aggregate Technological Knowledge Growth

The technological knowledge embodied in a new machine is non-rival. Hence, it must be temporarily excludable so that an investing firm can reap the economic benefits of its investment. I capture these features with the assumption that the technological knowledge embodied in a new machine is proprietary knowledge of an investing firm only in t, i. e., in the period when the investment is made. This may reflect patent protection, secrecy, or the required time competitors need to reverse-engineer a new machine and to understand the technology it uses.

In the periods after an automation investment, the new knowledge associated with it becomes part of the indicator A_t, A_{t+1},..., with no further scope for proprietary exploitation. The evolution of this indicator is given by

$$A_t = \max_{n \in [0,N_t]} \{a_t(n)\} = A_{t-1} \max_{n \in [0,N_t]} \{1 + q_t(n)\}. \tag{2.7}$$

Accordingly, the stock of technological knowledge to which all firms have access at the beginning of period $t+1$ reflects the highest level of technological knowledge attained for any of the $n \in [0, N_t]$ tasks performed at t.

Observe that non-rivalry and the limited excludability provide the link between the notions of embodied technological knowledge at the level of each automation investment and disembodied knowledge accumulation at the level of the economy as a whole. As suggested by Figure 2.1, this link will be the source of technological progress and sustained economic growth.

2.1.3 The Profit-Maximizing Production Plan

The representative firm takes the sequence $\{w_t, R_t, A_{t-1}\}_{t=1}^{\infty}$ of real wages, real rental rates of capital, and the aggregate productivity indicators as given and chooses a production plan

$$\left(Y_t, K_t, I_t, N_t, H_t^d, q_t(n), h_t(n), i\left(q_t(n)\right)\right)$$

for all $n \in [0, N_t]$ and all t. Here, K_t is the aggregate demand for fixed capital, I_t the aggregate demand for automation investments, and H_t^d the aggregate demand for hours worked, i. e.,

$$I_t = \int_0^{N_t} i\left(q_t(n)\right) dn \quad \text{and} \quad H_t^d = \int_0^{N_t} h_t(n) dn.$$

The optimal production plan maximizes the sum of the present discounted values of profits in all periods. Since an automation investment generates proprietary technological knowledge only in the period when it is made, the inter-temporal maximization boils down to the maximization of per-period profits denoted by Π_t.

In view of (2.2) and (2.3) the time spent on the performance of task n is

$$h_t(n) = \frac{1}{A_{t-1}\left(1 + q_t(n)\right)}. \tag{2.8}$$

Hence, task n gives rise to a wage cost, $w_t h_t(n)$, and an investment cost, $i(q_t(n))$. Let $c_t(n)$ denote these costs, i. e.,

$$c_t(n) = \frac{w_t}{A_{t-1}\left(1 + q_t(n)\right)} + i\left(q_t(n)\right). \tag{2.9}$$

Accordingly, for each period t, the firm's optimal plan solves

$$\max_{\left(K_t, N_t, [q_t(n)]_{n=0}^{n=N_t}\right)} \qquad \Pi_t = \Gamma K_t^{\gamma} N_t^{1-\gamma} - R_t K_t - \int_0^{N_t} c_t(n) dn.$$

Here, the last term is the sum of the costs of all performed tasks.

At all t, the firm's maximization problem may be split up into two parts. First, for each $n \in \mathbb{R}_+$ the firm chooses the value $q_t(n) \in \mathbb{R}_+$ that minimizes the cost of task n, i. e., it solves

$$\min_{[q_t(n)]_{n=0}^{\infty}} c_t(n). \tag{2.10}$$

Second, at minimized costs per task, the firm determines the profit-maximizing number of tasks, N_t, and the desired amount of fixed capital, K_t.

Cost Minimization per Task

Let $\omega_t \equiv w_t / A_{t-1}$ denote the wage cost of tasks at t before an automation investment is undertaken. Then, for all $n \in \mathbb{R}_+$ the respective first-order (sufficient) condition to problem (2.10) is

$$\frac{-\omega_t}{(1+q_t(n))^2} + \alpha \geq 0, \quad \text{with strict inequality only if } q_t(n) = 0. \tag{2.11}$$

This condition relates the marginal reduction of task n's wage cost to the marginal increase in its investment cost. Since this trade-off is the same for all tasks we have $q_t(n) = q_t$ where

$$q_t = q\left(\omega_t\right) \equiv \begin{cases} -1 + \sqrt{\frac{\omega_t}{\alpha}} & \text{if } \omega_t \geq \alpha, \\ \\ 0 & \text{if } \omega_t \leq \alpha. \end{cases} \tag{2.12}$$

Hence, if the wage cost per working hour under the old technology is greater than the marginal investment outlays at $q_t = 0$, i. e., if $\omega_t > \alpha$, then $q_t > 0$ with $\partial q\left(\omega_t\right) / \partial \omega_t > 0$. In other words, the more expensive the working hour under the old technology is expected to be the higher is q_t.

If $\omega_t \leq \alpha$ then no automation investments are undertaken and the performance of tasks occurs with old machines that embody the technology represented by A_{t-1}. Intuitively, this corner solution arises if at $q_t(n) = 0$ the marginal reduction of the wage cost is too small compared to the marginal investment cost, $\alpha > 0$. Then, labor is so cheap that it retains its comparative advantage over new machines.

Using (2.12) in (2.4), (2.8), and (2.9) delivers the cost-minimizing choices per task of hours worked, investment outlays, and costs denoted by h_t, i_t, and c_t. For further reference, define

$$h\left(\omega_t\right) \equiv \frac{1}{1+q\left(\omega_t\right)}, \quad i\left(\omega_t\right) \equiv \alpha q\left(\omega_t\right), \quad \textit{and} \quad c\left(\omega_t\right) \equiv \omega_t h\left(\omega_t\right) + i\left(\omega_t\right).$$

Proposition 2.1 *(Cost Minimization per Task)*

The minimization of costs per task delivers continuous, piecewise defined functions

$$q_t = q(\omega_t), \quad h_t = \frac{h(\omega_t)}{A_{t-1}}, \quad i_t = i(\omega_t), \quad \text{and} \quad c_t = c(\omega_t).$$

In addition to (2.12) the following closed-form solutions obtain:

- *if* $\omega_t \geq \alpha$ *then*

$$h_t = \frac{1}{A_{t-1}}\sqrt{\frac{\alpha}{\omega_t}}, \quad i_t = \sqrt{\alpha\omega_t} - \alpha, \quad \text{and} \quad c_t = 2\sqrt{\alpha\omega_t} - \alpha,$$

- *if* $\omega_t \leq \alpha$ *then*

$$h_t = \frac{1}{A_{t-1}}, \quad i_t = 0, \quad \text{and} \quad c_t = \omega_t.$$

The following corollary to Proposition 2.1 highlights that cost-minimizing automation investments give rise to a *rationalization effect* and to a *productivity effect*.

Corollary 2.1 *(Rationalization and Productivity Effect)*

If $\omega_t > \alpha$ *then*

$$h_t < \frac{1}{A_{t-1}} \quad (\textit{rationalization effect})$$

and

$$c_t < \omega_t \quad (\textit{productivity effect}).$$

Hence, if automation is profitable then it means rationalization, i. e., fewer working hours per task. The productivity effect results in spite of investment outlays since a cost-minimizing automation investment reduces the overall cost per task.

A higher wage strengthens the rationalization and the productivity effect. For the former, this holds since $\partial q/\partial\omega_t > 0$ implies $dh_t/d\omega_t = (\partial h/\partial\omega_t)/A_{t-1} < 0$. For the latter, this is true since $dc_t/d\omega_t = h(\omega_t) \in (0,1)$ so that both the difference $\omega_t - c_t$ and the ratio ω_t/c_t increase in ω_t.

Profit-Maximization at Minimized Costs

At minimized costs per task profits at t become

$$\Pi_t = \Gamma K_t^{\gamma} N_t^{1-\gamma} - R_t K_t - c_t N_t,$$

and the maximization with respect to N_t and K_t delivers the first-order conditions

$$N_t: \quad \Gamma(1-\gamma) K_t^{\gamma} N_t^{-\gamma} - c_t = 0 \quad \text{and} \quad K_t: \quad \Gamma\gamma K_t^{\gamma-1} N_t^{1-\gamma} - R_t = 0. \tag{2.13}$$

Both conditions require the respective value product to equal marginal cost. The marginal cost of task N_t is c_t. This leads to the following proposition where

$$N(c_t) \equiv \left(\frac{\Gamma(1-\gamma)}{c_t}\right)^{\frac{1}{\gamma}} \quad \text{and} \quad Y(c_t) \equiv \Gamma\left(\frac{\Gamma(1-\gamma)}{c_t}\right)^{\frac{1-\gamma}{\gamma}}.$$

Proposition 2.2 *(Profit-Maximizing Tasks, Output, Profits, and the Factor-Price Frontier)*

Given K_t, the profit-maximizing amounts of tasks and output at t are

$$N_t = K_t N(c_t) \quad \textit{and} \quad Y_t = K_t Y(c_t).$$

Moreover, the factor-price frontier is $R_t = \gamma Y(c_t)$ and $\Pi_t = 0$.

Hence, given K_t, the profit-maximizing levels of tasks and output may be expressed as functions of c_t. With a slight abuse of notation, I shall henceforth denote these levels by N_t and Y_t. The functions $N(c_t)$ and $Y(c_t)$ show, respectively, how the amount of tasks per unit of fixed capital, N_t/K_t, and the productivity of fixed capital, Y_t/K_t, hinges on the minimized costs per task, c_t. A decline in c_t increases the profit-maximizing amount of tasks since the marginal value product of tasks is equal to a lower cost per task at a greater N_t. Hence, $N'(c_t) < 0$. As $Y(c_t) = \Gamma N(c_t)^{1-\gamma}$ this implies $Y'(c_t) < 0$. Moreover, the factor-price frontier dictates that R_t will fall in c_t, too. Finally, constant returns to scale of the production function imply $\Pi_t = 0$.

The following corollary to Proposition 2.2 establishes that automation gives rise to a *task expansion effect* and an *output expansion effect*.

Corollary 2.2 *(Task and Output Expansion Effect)*

If $\omega_t > \alpha$ then

$$N(c_t) > N(\omega_t) \quad \textit{(task expansion effect)}$$

and

$$Y(c_t) > Y(\omega_t) \quad \textit{(output expansion effect).}$$

If $\omega_t > \alpha$ then firms undertake automation investments and the productivity effect of Corollary 2.1 implies $c_t < \omega_t$. Then, the *task expansion effect* and the *output expansion effect* of automation follow since $N'(c_t) < 0$ and $Y'(c_t) < 0$.

Finally, let me note that the profit-maximizing production plan determines the aggregate demands for fixed capital, K_t, automation investments, $I_t = i_t N_t$, and for hours worked $H_t^d = h_t N_t$. Moreover, unlike in Acemoglu and Restrepo (2018d) and Zeira (1998), automation appears as labor-augmenting technical change in the aggregate production function (2.1). To see this, observe that cost minimization implies $q(n_t) = q_t$ so that the production function of each task is $1 = A_{t-1}(1+q_t)\,h_t$. Here, the rationalization effect means that automation is labor-saving in the sense that fewer hours of labor are needed in the performance of a task, i. e., a higher q_t means a lower h_t. However, if N_t tasks are performed then $N_t = N_t A_{t-1}(1+q_t)\,h_t = A_{t-1}(1+q_t)\,H_t^d$ as $H_t^d = h_t N_t$. Hence,[10] with (2.1)

$$Y_t = \Gamma K_t^{\gamma}\left(A_{t-1}(1+q_t)\,H_t^d\right)^{1-\gamma}.$$

2.1.4 Automation and the Labor Share

Since $\Pi_t = 0$, the economy satisfies $Y_t - R_t K_t - c_t N_t = Y_t - R_t K_t - w_t h_t N_t - i_t N_t = 0$. Let GDP_t denote the economy's net output at t, i. e., $GDP_t = Y_t - i_t N_t$. Then, total earned income satisfies $R_t K_t + w_t h_t N_t = GDP_t$, and the labor share is defined as

$$LS_t \equiv \frac{w_t h_t N_t}{GDP_t}. \tag{2.14}$$

Proposition 2.3 *(Automation and the Labor Share)*

If $\omega_t > \alpha$ then

$$LS_t = (1-\gamma)\left(\frac{w_t h_t}{w_t h_t + \gamma i_t}\right).$$

Hence, automation unequivocally reduces the labor share because it involves investment outlays, $i_t > 0$. Since the labor and the capital share, $R_t K_t / GDP_t$, add up to one, automation will increase the latter.[11]

The intuition for Proposition 2.3 is as follows. Without automation investments net and gross output coincide. Then, the production function (2.1) implies that the share of tasks in GDP, $c_t N_t / Y_t$, is equal to $1-\gamma$. Moreover, since $c_t = w_t$ and $h_t = 1$ the share of tasks

[10] This finding uses the minimization of costs per task and the definition of the firm's demand for hours worked. Therefore, it generalizes beyond the Cobb-Douglas form to any production function $F(K_t, N_t)$. For these functions one obtains $Y_t = F\left(K_t, A_{t-1}(1+q_t)\,H_t^d\right)$. Hence, the term labor-augmenting technical change is indeed meaningful here.

[11] Here, investment outlays are treated as a flow input. Therefore, no income accrues to machines. However, one readily verifies that the qualitative results of Proposition 2.3 and Corollary 2.3 below remain unchanged if new machines are treated as a stock and deliver rental income to their owners.

coincides with the labor share. With automation investments the share of tasks in GDP is

$$\frac{c_t N_t}{Y_t - i_t N_t} = (1-\gamma)\left(\frac{c_t}{w_t h_t + \gamma i_t}\right),$$

where use is made of Proposition 2.1 and 2.2. This share is split up into the labor share, LS_t, and the "share of automation investments," $i_t N_t / GDP_t = (1-\gamma)\left(i_t/(w_t h_t + \gamma i_t)\right)$.

The following corollary shows that the decline in the labor share, LS_t, is more pronounced the stronger the incentives to automate.

Corollary 2.3 *(Automation Incentives and the Lower Bound of the Labor Share)*

If $\omega_t > \alpha$ then it holds that

$$\frac{\partial LS_t}{\partial \omega_t} < 0.$$

Moreover,

$$\lim_{\omega_t \to \infty} LS_t = \frac{1-\gamma}{1+\gamma}.$$

Hence, a higher expected wage induces more automation and reduces the labor share. However, the labor share remains strictly positive even if the incentives to automate become very strong and, asymptotically, $h_t \to 0$. These findings obtain since for the chosen functional forms the labor share may be written as

$$LS_t = (1-\gamma)\left(\frac{1}{1+\gamma\left(\frac{i_t}{w_t h_t}\right)}\right) = (1-\gamma)\left(\frac{1}{1+\gamma\left(1-\sqrt{\frac{\alpha}{\omega_t}}\right)}\right), \tag{2.15}$$

i. e., it depends on the per-task amount of investment outlays in wage costs, $i_t/(w_t h_t)$. This ratio is smaller than unity, increases in ω_t, and converges to 1.

2.2 The Household Sector

Individuals live for possibly two periods, young and old age. When young, they supply labor, earn wage income, enjoy leisure and consumption, and save. At the onset of old age, they face a survival probability $\mu \in (0,1)$. Surviving old individuals retire and consume their wealth.[12]

[12] Hence, by assumption aging may affect the endogenous labor supply when young but not the timing of retirement. To a first approximation, this does not seem too far from reality. For instance, Bloom, Canning,

The population at t consists of L_t young (cohort t) and μL_{t-1} old individuals. Due to birth and other demographic factors the number of young individuals between two adjacent periods grows at rate $g_L > (-1)$. For short, I shall refer to g_L as the fertility rate.

Population aging is the result of an increase in life expectancy and/or a decline in fertility. An increase in μ and/or a decline in g_L capture this. These parameter changes also lift the old-age dependency ratio. For period t the latter is defined as

$$OADR_t \equiv \frac{\mu L_{t-1}}{L_t} = \frac{\mu}{1+g_L}. \tag{2.16}$$

Hence, $OADR_t$ is determined by the survival probability and the fertility rate of cohort $t-1$. There is population aging between period $t-1$ and t if $OADR_t > OADR_{t-1}$. Accordingly, an increase in the survival probability of cohort $t-1$ and/or a decline in the fertility rate of this cohort implies population aging.

For cohort t, denote consumption when young and old by c_t^y and c_{t+1}^o, and leisure time enjoyed when young by l_t. The periodic time endowment is normalized to unity. Then, $l_t = 1 - h_t^s$, where $h_t^s \in [0,1]$ is working hours supplied by cohort t when young.

Individuals of all cohorts assess bundles $\left(c_t^y, l_t, c_{t+1}^o\right)$ according to an expected lifetime utility function, U, featuring a periodic utility function of the generalized log-log type proposed by Boppart and Krusell (2020). The utility after death is set equal to zero. Accounting for retirement when old, i. e., $l_{t+1} = 1$, cohort t's expected utility is

$$U\left(c_t^y, l_t, c_{t+1}^o\right) = \ln c_t^y + \ln\left(1 - \phi\left(1-l_t\right)\left(c_t^y\right)^{\frac{\nu}{1-\nu}}\right) + \mu\beta \ln c_{t+1}^o, \tag{2.17}$$

where $0 < \beta < 1$ is the discount factor, $\phi > 0$ and $\nu \in (0,1)$. For ease of notation, I use henceforth $x_t \equiv \left(1-l_t\right)\left(c_t^y\right)^{\frac{\nu}{1-\nu}}$.

The term $\ln\left(1-\phi x_t\right)$ reflects the disutility of labor when young. The parameter ϕ captures characteristics of the labor market that affect the disutility of labor in the population irrespective of the amount of hours worked and the level of consumption. These include, e. g., the level of occupational safety regulations and the climatic conditions under which labor is done (Landes (1998)). As shown in Irmen (2018b), $\nu \in (0,1)$ assures that consumption and leisure are complements in the cardinal sense of $\partial^2 U/\partial c_t^y \partial l_t > 0$.

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and Fink (2010), p. 5-6, report for a sample of 43 mostly developed countries that the average male life expectancy increased between 1965 and 2005 by 8.8 years whereas the average legal male retirement age increased by less than half a year. More strikingly, the correlation between the change in male life expectancy and the change in the retirement age over this time-span is small and negative. While recent years have seen political initiatives to increase the statutory retirement age, e. g., in the EU-27, there is often substantial political resistance (see, e. g., New York Times (2011) or New York Times (2019) on France). Whether and how such changes impact on the effective retirement age that people choose is likely to depend on the future evolution of life expectancy and on institutional details of the retirement scheme (Gruber and Wise (2004)). I shall get back to this issue in Section 6.

Expected utility, U, is strictly monotone and strictly concave if

$$1 - 2\nu - (1-\nu)\phi x_t > 0. \tag{2.18}$$

This condition requires $\nu < 1/2$. Henceforth, I refer to the set of bundles $\left(c_t^y, l_t, c_{t+1}^o\right) \in \mathbb{R}_{++} \times [0,1] \times \mathbb{R}_{++}$ that satisfy (2.18) as the set of permissible bundles denoted by $\mathcal{P}$.

At the end of their young age, individuals of cohort t deposit their entire savings with life insurers in exchange for annuity policies. These insurers rent the savings out as fixed capital to the firms producing in $t+1$. In return, the latter pay a (perfect foresight) real rental rate R_{t+1} per unit of savings. Perfect competition among risk-neutral life insurers guarantees a gross return to a surviving old at $t+1$ of R_{t+1}/μ. Hence, cohort t faces the periodic budget constraints

$$c_t^y + s_t \leq w_t(1-l_t) \quad \text{and} \quad c_{t+1}^o \leq \frac{R_{t+1}}{\mu} s_t. \tag{2.19}$$

I refer to $\left(c_t^y, l_t, c_{t+1}^o, s_t, h_t^s\right)$ as a plan of cohort t. The optimal plan solves

$$\max_{\left(c_t^y, l_t, c_{t+1}^o, s_t\right) \in \mathcal{P} \times \mathbb{R}} U\left(c_t^y, l_t, c_{t+1}^o\right) \quad \text{subject to (2.19)} \tag{2.20}$$

and includes the utility maximizing supply of working hours as $h_t^s = 1 - l_t$. Before I fully characterize the solution to this problem the following assumption must be introduced.

Assumption 1 *For all t it holds that*

$$w_t > w_c \equiv \left(\frac{(1+\mu\beta)(1-\nu)}{\left(\phi\left(1+(1+\mu\beta)(1-\nu)\right)\right)^{1-\nu}\left(1-\nu(1+\mu\beta)\right)^{\nu}} \right)^{\frac{1}{\nu}}$$

and

$$0 < \nu < \bar{\nu}(\mu\beta) \equiv \frac{3 + \mu\beta - \sqrt{5 + \mu\beta(2+\mu\beta)}}{2(1+\mu\beta)}.$$

As will become clear in the Proof of Proposition 2.4 below, Assumption 1 assures two things. First, if the real wage exceeds the critical level w_c then cohort t's demand for leisure is strictly positive. Second, the unique bundle identified by the Lagrangian associated with problem (2.20) satisfies condition (2.18).[13] Hence, it is a global maximum on the choice set $\mathcal{P} \times \mathbb{R}$.

[13]The function $\bar{\nu}(\mu\beta)$ is strictly positive and declining in $\mu\beta$ with $\bar{\nu}(0) \approx 0.382$ and $\bar{\nu}(1) \approx 0.293$. Hence, Assumption 1 imposes a tighter constraint on ν than just $\nu < 1/2$ which is necessary for (2.18) to hold.

Proposition 2.4 *(Optimal Plan of Cohort t)*

Suppose Assumption 1 holds. Then, the optimal plan of cohort $t = 1, 2, ..., \infty$ involves

$$h_t^s = w_c^\nu w_t^{-\nu}, \qquad c_t^y = \frac{1-\nu\left(1+\mu\beta\right)}{\left(1+\mu\beta\right)\left(1-\nu\right)} w_c^\nu w_t^{1-\nu},$$

$$s_t = \frac{\mu\beta}{\left(1+\mu\beta\right)\left(1-\nu\right)} w_c^\nu w_t^{1-\nu}, \qquad c_{t+1}^o = \frac{\beta R_{t+1}}{\left(1+\mu\beta\right)\left(1-\nu\right)} w_c^\nu w_t^{1-\nu}.$$

For surviving members of cohort 0*, consumption when old is $c_1^o = R_1 s_0/\mu > 0$ where $s_0 > 0$ is given.*

According to Proposition 2.4 cohort t's supply of hours worked declines in the wage with an elasticity equal to ν. As a consequence, the positive response of s_t, c_t^y, and c_{t+1}^o to a wage hike is less than proportionate. Observe that c_t^y and s_t may be expressed, respectively, as the product of a marginal (and average) propensity to consume or to save and the wage income, i. e.,

$$c_t^y = \frac{1-\nu(1+\mu\beta)}{(1+\mu\beta)(1-\nu)} w_t h_t^s \quad \text{and} \quad s_t = \frac{\mu\beta}{(1+\mu\beta)\left(1-\nu\right)} w_t h_t^s. \tag{2.21}$$

This helps to understand how a change in the life expectancy affects the optimal plan.

Corollary 2.4 *(Life-Expectancy and the Optimal Plan of Cohort t)*

If $w_t > w_c$ then it holds that

$$\frac{\partial h_t^s}{\partial \mu} > 0, \quad \frac{\partial c_t^y}{\partial \mu} < 0, \quad \frac{\partial s_t}{\partial \mu} > 0, \quad \frac{\partial c_{t+1}^o}{\partial \mu} < 0.$$

Hence, a higher life expectancy increases the supply of hours worked. This reflects the appreciation of the utility when old relative to the utility when young. Through this channel the demand for leisure declines and h_t^s increases.[14]

The effect of a higher life expectancy on consumption when young is the result of two opposing channels. On the one hand, for a given wage income, the propensity to consume in (2.21) falls. This reflects the desire to shift resources into the second period of life which now has more weight. On the other hand, there will be more income since the supply of hours worked increases. Then, consumption smoothing calls for more consumption when young. Overall, the former effect dominates so that c_t^y falls in μ.

[14]A higher μ also reduces the gross rate of return to a surviving old, R_{t+1}/μ. However, for U of (2.17) the substitution and the income effect associated with such a reduction on h_t^s, c_t^y, and s_t cancel out.

The same two channels determine the effect of a higher life expectancy on savings. However, now they are reinforcing. Indeed, for a given wage income, the propensity to save in (2.21) increases. Moreover, a higher wage income and consumption smoothing imply more savings, too. Hence, s_t increases in μ.

Finally, consumption when old declines with a higher life expectancy. Again, two channels of opposite sign are at work. On the one hand, savings increase pushing c^o_{t+1} upwards. On the other hand, the rate of return on savings for a surviving old, R_{t+1}/μ, falls. As the latter dominates, c^o_{t+1} declines in μ.

3 Inter-temporal General Equilibrium

3.1 Definition

A *price system* corresponds to a sequence $\{w_t, R_t\}_{t=1}^{\infty}$. An *allocation* is a sequence

$$\{c^y_t, l_t, c^o_t, s_t, h^s_t, Y_t, K_t, N_t, H^d_t, I_t, q_t(n), a_t(n), h_t(n), i\left(q_t(n)\right)\}_{t=1}^{\infty}$$

for all tasks $n \in [0, N_t]$. It comprises a plan $\{c^y_t, l_t, c^o_t, s_t, h^s_t\}_{t=1}^{\infty}$ for all cohorts, consumption of the old at $t = 1$, c^o_1, and a plan $\{Y_t, K_t, N_t, H^d_t, I_t, q_t(n), a_t(n), h_t(n), i\left(q_t(n)\right)\}_{t=1}^{\infty}$ for the production sector.

For an exogenous evolution of the labor force, $L_t = L_1\left(1 + g_L\right)^{t-1}$ with $L_1 > 0$ and $g_L > (-1)$, and initial levels of fixed capital, $K_1 > 0$, and technological knowledge, $A_0 > 0$, an *inter-temporal general equilibrium with perfect foresight* corresponds to a price system, an allocation, and a sequence $\{A_t\}_{t=1}^{\infty}$ of the aggregate technological knowledge indicator that comply with the following conditions for all $t = 1, 2, ..., \infty$:

(E1) The production sector satisfies Propositions 2.1 and 2.2.

(E2) The indicator A_t evolves according to (2.7).

(E3) The plan of each cohort satisfies Proposition 2.4.

(E4) The market for the manufactured good clears, i. e.,

$$\mu L_{t-1} c^o_t + L_t c^y_t + I_t + I^K_t = Y_t,$$

where I^K_t is aggregate investment in fixed capital.

(E5) There is full employment of labor, i. e.,

$$h_t N_t = h^s_t L_t.$$

(E1) assures the optimal behavior of the production sector and zero profits. In conjunction with (E2) the evolution of technological knowledge for the economy as a whole boils down to

$$A_t = a_t = A_{t-1}(1 + q_t), \quad \text{for all } t \text{ given } A_0 > 0. \tag{3.1}$$

(E3) guarantees the optimal behavior of the household sector under perfect foresight. Since the old own the capital stock, their consumption at $t = 1$ is $\mu L_0 c_1^o = R_1 K_1$ and $s_0 = K_1/L_0$. (E4) states that the aggregate demand for the manufactured good produced at t is equal to its supply. Aggregate demand at t comprises aggregate consumption, $\mu L_{t-1}c_t^o + L_t c_t^y$, aggregate automation investments, I_t, and aggregate investment in fixed capital, I_t^K. On the supply side, it reflects the (innocuous) assumption that fixed capital fully depreciates after one period. According to (E5) the aggregate demand for hours worked must be equal to its supply. Here, use is made of (E1) in that $h_t(n) = h_t$ for all n.

To focus the discussion I henceforth restrict attention to constellations where, for all t, profit-maximizing firms decide to automate and utility-maximizing cohorts express a strictly positive demand for leisure. The following assumption accomplishes this.

Assumption 2 *For all t it holds that*

$$w_t > \alpha A_{t-1} > w_c.$$

The inequality $w_t > \alpha A_{t-1}$ is equivalent to $\omega_t > \alpha$. Hence, automation investments are profit-maximizing (see Proposition 2.1). If $w_t > w_c$ then cohort t has a strictly positive demand for leisure (see Proposition 2.4). Finally, assuming $\alpha A_{t-1} > w_c$ for all t simplifies the analysis of the transitional dynamics since the distinction between the three regimes $\alpha A_{t-1} < w_c$, $\alpha A_{t-1} = w_c$, and $\alpha A_{t-1} > w_c$ can be neglected. As A_{t-1} grows over time $\alpha A_{t-1} > w_c$ is satisfied for all t if $A_0 > w_c/\alpha$.

3.2 Labor Market Equilibrium

With Proposition 2.1 and 2.2 the aggregate demand for hours worked at t may be expressed as

$$\begin{aligned} H_t^d = H^d(\omega_t) &\equiv \left(\frac{K_t}{A_{t-1}}\right) h(\omega_t) N(c_t) \\ &= \left(\frac{K_t}{A_{t-1}}\right) \sqrt{\frac{\alpha}{\omega_t}} \left(\frac{\Gamma(1-\gamma)}{c_t}\right)^{\frac{1}{\gamma}} \end{aligned} \tag{3.2}$$

with slope

$$\frac{dH^d(\omega_t)}{d\omega_t} = \underbrace{\left(\frac{K_t}{A_{t-1}}\right) N(c_t) \underbrace{\frac{\partial h(\omega_t)}{\partial \omega_t}}_{(-)}}_{\text{Aggregate Rationalization Effect } (-)} + \underbrace{\left(\frac{K_t}{A_{t-1}}\right) h(\omega_t) \underbrace{\frac{\partial N(c_t)}{\partial c_t}}_{(-)} \underbrace{\frac{\partial c(\omega_t)}{\partial \omega_t}}_{(+)}}_{\text{Aggregate Task Expansion Effect } (-)} < 0.$$

An increase in w_t, respectively ω_t, induces an *aggregate rationalization effect* and an *aggregate task expansion effect*. Both effects are negative. The former means that fewer working hours per task will be demanded for all performed tasks. The latter reflects the increase in the cost per task and the concomitant decline in the total number of performed tasks for given working hours per task.

From Proposition 2.4 the aggregate supply of hours worked at t for $w_t > w_c$ is $H_t^s = w_c^{\nu} w_t^{-\nu} L_t$ and may be expressed in terms of ω_t as

$$H_t^s = \left(\frac{w_c}{A_{t-1}}\right)^{\nu} \omega_t^{-\nu} L_t \equiv H^s\left(\omega_t, L_t, w_c\right). \tag{3.3}$$

Then, the equilibrium wage, $\hat{w}_t$, and the corresponding level of employment, $\hat{H}_t$, are determined by the labor market equilibrium condition

$$H^d\left(\hat{\omega}_t\right) = H^s\left(\hat{\omega}_t, L_t, w_c\right), \tag{3.4}$$

where $\hat{\omega}_t \equiv \hat{w}_t / A_{t-1}$.

Since both, the aggregate demand for and the aggregate supply of hours worked, fall in the real wage there may be none, one, or multiple wage levels at which demand is equal to supply. This section establishes the existence of a unique labor market equilibrium that satisfies Assumption 2. To accomplish this it proves useful to introduce the following notation:

$$k_t \equiv \frac{K_t}{A_{t-1}^{1-\nu} L_t}, \quad \text{and} \quad \underline{k}_c \equiv w_c^{\nu} \alpha^{-\nu} \left[\frac{\alpha}{\Gamma(1-\gamma)}\right]^{\frac{1}{\gamma}}.$$

Henceforth, I shall refer to k_t as the *efficient capital intensity*. It will later serve as the state variable of the dynamical system. The parameter $\underline{k}_c$ denotes a critical level of the efficient capital intensity to be interpreted below.

Proposition 3.1 *(Labor Market Equilibrium)*

Suppose $\alpha A_{t-1} > w_c$ holds. Then, a unique labor market equilibrium $\left(\hat{w}_t, \hat{H}_t\right)$ with $\hat{w}_t > \alpha A_{t-1}$ exists for all $t = 1, 2, ..., \infty$ if and only if

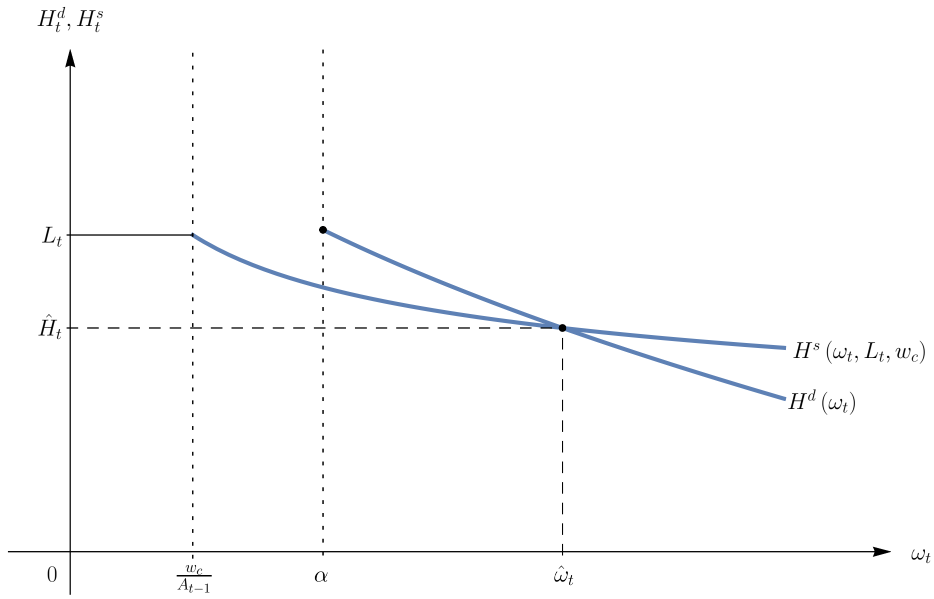
$$k_t > \underline{k}_c.$$

Moreover, the labor market equilibrium defines a function $\hat{\omega} : (\underline{k}_c, \infty) \to (\alpha, \infty)$ such that

$$\hat{\omega}_t = \omega\left(k_t\right) \quad \textit{with} \quad \omega'\left(k_t\right) > 0.$$

Proposition 3.1 makes two points. First, it establishes the existence of a unique labor market equilibrium consistent with Assumption 2 if k_t is sufficiently large. Intuitively, this follows since the aggregate supply of hours worked, $H^s\left(\omega_t, L_t, w_c\right)$, is sufficiently

Figure 3.1: The Labor-Market Equilibrium.



Note: Assumption 2 is equivalent to $\omega_t > \alpha > w_c/A_{t-1}$. The labor market equilibrium $(\hat{\omega}_t, \hat{H}_t)$ satisfies $\hat{H}_t = H^d(\hat{\omega}_t) = H^s(\hat{\omega}_t, L_t, w_c) < L_t$. A unique equilibrium exists since $H^s(\omega_t, L_t, w_c)$ is sufficiently flatter than $H^d(\omega_t)$ and $H^d(\alpha) > H^s(\alpha, L_t, w_c)$.

flatter than the aggregate demand, $H^d(\omega_t)$. Moreover, $k_t > \underline{k}_c$ assures that $H^d(\alpha) > H^s(\alpha, L_t, w_c)$. Accordingly, $\hat{\omega}_t > \alpha$ and the equilibrium wage satisfies $\hat{w}_t > \alpha A_{t-1}$ (see Figure 3.1 for an illustration).

Second, it lays open that $\hat{\omega}_t$ can be expressed as a function of k_t. Indeed, with (3.2) and (3.3) the labor market equilibrium condition (3.4) may be stated as

$$k_t = \frac{w_c^\nu \hat{\omega}_t^{-\nu}}{h(\hat{\omega}_t) N(c(\hat{\omega}_t))} = w_c^\nu \hat{\omega}_t^{-\nu} \sqrt{\frac{\hat{\omega}_t}{\alpha}} \left(\frac{2\sqrt{\alpha \hat{\omega}_t} - \alpha}{\Gamma(1-\gamma)} \right)^{\frac{1}{\gamma}}. \tag{3.5}$$

The right-hand side captures the effect of ω_t on the supply relative to the demand of hours worked. If $\omega_t = \alpha$ then (3.5) boils down to $k_t = \underline{k}_c$. For $\omega_t \geq \alpha$ the right-hand side is monotonically increasing in ω_t and becomes unbounded as $\omega_t \to \infty$. Hence, a higher w_t induces a proportionate decline in the demand for hours worked that dominates the proportionate decline in the supply. Moreover, for any $k_t > \underline{k}_c$ there is a unique $\hat{\omega}_t = \omega(k_t) > \alpha$. The derivative $\omega'(k_t) > 0$ captures that a greater K_t increases H_t^d whereas a lower L_t reduces H_t^s. Moreover, the effect of a lower A_{t-1} on $H^d(\hat{\omega}_t)$ is stronger than the one on $H^s(\hat{\omega}_t, L_t, w_c)$ so that $\hat{\omega}_t$ increases.

3.3 Dynamical System

The transitional dynamics of the inter-temporal general equilibrium can be analyzed through the evolution of a single state variable, k_t. To derive the equilibrium sequence $\{k_t\}_{t=1}^{\infty}$ observe that conditions (E3) and (E4) require investments in fixed capital to equal savings, i. e., $I_t^K = s_t L_t = K_{t+1}$, or

$$\frac{\mu\beta}{(1+\mu\beta)(1-\nu)} w_t h_t^s L_t = K_{t+1}, \quad \text{for all } t = 1, 2, ..., \infty. \tag{3.6}$$

Using Proposition 2.4 the latter equation may be expressed as

$$\Omega \omega_t^{1-\nu} = k_{t+1}, \quad \text{for all } t = 1, 2, ..., \infty, \tag{3.7}$$

where,

$$\Omega \equiv \frac{\mu\beta w_c^{\nu}}{(1+\mu\beta)(1-\nu)(1+g_L)}$$

summarizes preference and demography parameters that affect the relationship between ω_t and k_{t+1}. Henceforth, I shall refer to equation (3.7) as the capital market equilibrium condition. The equilibrium difference equation that describes the evolution of k_t results from replacing ω_t of (3.7) with the labor market clearing condition $\hat{\omega}_t = \omega(k_t)$ of Proposition 3.1. This gives

$$k_{t+1} = \Omega \left[\omega(k_t)\right]^{1-\nu}. \tag{3.8}$$

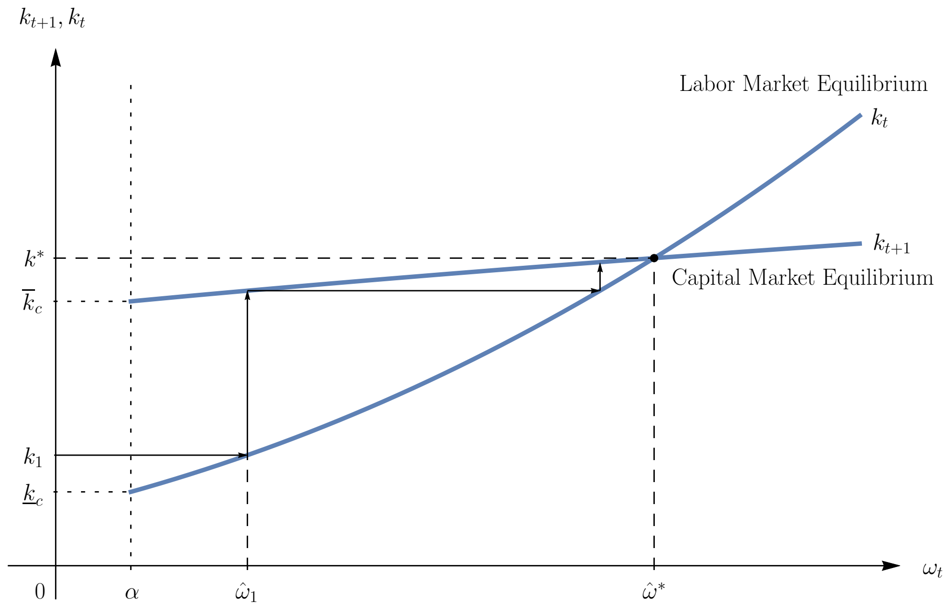
Hence, k_{t+1} is increasing in k_t and greater than $\Omega\left[\omega(\underline{k}_c)\right]^{1-\nu} = \Omega\alpha^{1-\nu} \equiv \overline{k}_c$, which is the lowest permissible level of savings at t per efficient worker in $t+1$, $A_t^{1-\nu} L_{t+1}$. For the labor market to satisfy $\hat{\omega}_t > \alpha$ for all t, $k_t > \underline{k}_c$ is required for all t. Hence, (3.8) is to deliver a value $k_{t+1} > \underline{k}_c$ for all t. Then, the condition $\overline{k}_c > \underline{k}_c$ ensures for any $k_t > \underline{k}_c$ that $k_{t+1} > \underline{k}_c$ so that the labor market equilibrium at $t+1$ satisfies $\hat{\omega}_{t+1} > \alpha$.

Proposition 3.2 *(Dynamical System, Steady-State, and Transitional Dynamics)*

Let $\overline{k}_c > \underline{k}_c$ and consider initial values $(K_1, L_1, A_0) > 0$ such that $A_0 > w_c/\alpha$ and $k_1 > \underline{k}_c$. Then, there is a unique monotonic equilibrium sequence, $\{k_t\}_{t=1}^{\infty}$, governed by the equilibrium difference equation (3.8) with $k_t > \underline{k}_c$ for all t. Moreover, for any $k_1 > \underline{k}_c$ it holds that $\lim_{t\to\infty} k_t = k^* > \overline{k}_c$ where $k^* = \Omega\left[\omega(k^*)\right]^{1-\nu}$.

An instructive intuition for Proposition 3.2 can be gained from Figure 3.2. This figure depicts the labor market equilibrium at t of equation (3.5) and the capital market equilibrium at t of equation (3.7) for $\overline{k}_c > \underline{k}_c$. Then, for any $k_1 > \underline{k}_c$ the labor market at $t = 1$ delivers a unique $\omega_1 = \hat{\omega}_1 > \alpha$. Using $\hat{\omega}_1$ in the capital market equilibrium condition at $t = 1$ gives a unique $k_2 > \underline{k}_c$. Clearly, these steps apply to any pair $(k_t, k_{t+1}) > \underline{k}_c$. Figure 3.2 also highlights that the evolution of k_t will be monotonic with convergence to the steady state, $k^* > \overline{k}_c$.

Figure 3.2: The Dynamical System, Steady State, and Transitional Dynamics.



Note: For any $k_1 > \underline{k}_c$ the labor market at $t = 1$ delivers $\omega_1 = \hat{\omega}_1 > \alpha$. Since $\overline{k}_c > \underline{k}_c$, using $\hat{\omega}_1$ in the capital market equilibrium condition for $t = 1$ delivers $k_2 > \underline{k}_c$ and so forth.

4 Short-Run Macroeconomic Implications of Population Aging

Suppose cohort t anticipates an increase in its life expectancy and/or reduces its fertility. The short-run macroeconomic implications of an anticipated increase in life expectancy for automation, growth, and factor shares are contemporaneous whereas those of a decline in fertility materialize only in period $t + 1$.

Henceforth, I denote variables evaluated at the labor-market equilibrium with a hat, e. g., the cost-minimizing productivity growth rate of (2.12) evaluated at the labor-market equilibrium at t becomes $\hat{q}_t = q(\hat{\omega}_t)$.

4.1 Increasing Longevity

The following corollary to Proposition 3.1 shows how a change in μ affects the equilibrium wage and the labor productivity per hour worked.

Corollary 4.1 *(Short-Run Effects of μ: Equilibrium Wage and Productivity Growth per Hour Worked)*

Consider the labor market equilibrium of Proposition 3.1. Given k_t, it holds that

$$\frac{d\hat{w}_t}{d\mu} < 0 \quad \textit{and} \quad \frac{d\hat{q}_t}{d\mu} < 0.$$

Intuitively, if cohort t expects to live longer, then, in accordance with Corollary 2.4, the individual, hence, the aggregate supply of hours worked increases at the intensive margin. This reduces $\hat{\omega}_t$ and $\hat{w}_t$ since the aggregate demand for hours worked is steeper than the aggregate supply.[15] Analytically, total differentiation of the labor market equilibrium (3.4) uncovers this as

$$\frac{d\hat{\omega}_t}{dw_c} = \frac{\overbrace{\frac{\partial H^s\left(\hat{\omega}_t, L_t, w_c\right)}{\partial w_c}}^{(+)}}{\underbrace{\frac{\partial H^d\left(\hat{\omega}_t\right)}{\partial \omega_t} - \frac{\partial H^s\left(\hat{\omega}_t, L_t, w_c\right)}{\partial \omega_t}}_{(-)}} < 0,$$

where the denominator is negative since $\partial H^d\left(\hat{\omega}_t\right)/\partial \omega_t < \partial H^s\left(\hat{\omega}_t, L_t, w_c\right)/\partial \omega_t < 0$. Hence,

$$\frac{d\hat{\omega}_t}{d\mu} = \underbrace{\frac{d\hat{\omega}_t}{dw_c}}_{(-)}\underbrace{\frac{\partial w_c}{\partial \mu}}_{(+)} < 0.$$

To grasp the role of automation and of the elastic supply of hours worked for the effect of μ on $\hat{\omega}_t$ consider the labor market equilibrium condition as stated in (3.5). Here, total differentiation reveals that

$$\frac{d\hat{\omega}_t}{d\mu}\frac{\mu}{\hat{\omega}_t} = \frac{\overbrace{\frac{\partial w_c^{\nu}}{\partial \mu}\frac{\mu}{w_c^{\nu}}}^{(+)}}{\underbrace{\frac{\hat{\omega}_t}{h\left(\hat{\omega}_t\right)}\frac{\partial h\left(\hat{\omega}_t\right)}{\partial \omega_t} + \frac{\hat{\omega}_t}{N(\hat{c}_t)}\frac{\partial N(\hat{c}_t)}{\partial c_t}\frac{\partial c\left(\hat{\omega}_t\right)}{\partial \omega_t} + \nu}_{(-)}} < 0. \tag{4.1}$$

Hence, the negative sign of $d\hat{\omega}_t/d\mu$ obtains since the reinforcing impact of a higher $\hat{\omega}_t$ on the rationalization effect, represented by $\partial h\left(\hat{\omega}_t\right)/\partial \omega_t < 0$, and on the task expansion effect, represented by $\left(\partial N(\hat{c}_t)/\partial c_t\right)\left(\partial c\left(\hat{\omega}_t\right)/\partial \omega_t\right) < 0$, dominates the effect on the labor supply, represented by ν.

Let me denote the functional relationship between $\hat{\omega}_t$ and μ implied by the labor market equilibrium (3.4) by $\hat{\omega}_t = \hat{\omega}(\mu)$. Then, $\hat{q}_t = q\left(\hat{\omega}(\mu)\right)$ and an anticipated increase in longevity reduces the growth rate of the labor productivity per hour worked as

$$\frac{d\hat{q}_t}{d\mu} = \underbrace{\frac{\partial q\left(\hat{\omega}_t\right)}{\partial \omega_t}}_{(+)}\underbrace{\frac{d\hat{\omega}_t}{d\mu}}_{(-)} < 0.$$

[15] A higher μ means a higher w_c. In Figure 3.1 this shifts the aggregate supply of hours worked rightwards (not shown) so that $\hat{\omega}_t$ falls.

Given k_t, the equilibrium amount of performed tasks is $\hat{N}_t = A_{t-1}(1+q(\hat{\omega}_t))\hat{H}_t$. Accordingly, short-run GDP in absolute and per-capita terms is, respectively, $G\hat{D}P_t = F(K_t, \hat{N}_t) - \hat{N}_t i(\hat{\omega}_t)$ and $g\hat{d}p_t = G\hat{D}P_t / P_t$, where P_t is the population at t. Through the labor market equilibrium, i. e., $\hat{\omega}_t = \hat{\omega}(\mu)$, $G\hat{D}P_t$ and $g\hat{d}p_t$ hinge on μ.

Corollary 4.2 *(Short-Run Effects of μ: GDP and gdp)*

Consider the labor market equilibrium of Proposition 3.1. Given k_t, it holds that

$$\frac{dG\hat{D}P_t}{d\mu} > 0 \quad \textit{and} \quad \frac{dg\hat{d}p_t}{d\mu} > 0.$$

Hence, an anticipated increase in longevity increases GDP and gdp in the short run. The intuition is the following. A higher μ expands the labor supply and lowers the equilibrium wage. This affects $G\hat{D}P_t$ through two channels. First, given $\hat{H}_t$, the incentive to automate weakens. However, as firms choose the degree of automation per task and the number of tasks to maximize profits, this channel has no first-order effect on $G\hat{D}P_t$. Second, given $\hat{q}_t$, the decline in the equilibrium wage increases the level of employment. Accordingly, more tasks will be performed. Each of these additional tasks is associated with a strictly positive net output. Hence, $G\hat{D}P_t$ increases. Moreover, since the increase in μ does not affect the population size $g\hat{d}p_t$ increases, too.

Finally, denote the labor share evaluated at the labor-market equilibrium by $\hat{LS}_t = LS(\hat{\omega}_t)$ where $\hat{\omega}_t = \hat{\omega}(\mu)$.

Corollary 4.3 *(Short-Run Effects of μ: Labor Share)*

Consider the labor market equilibrium of Proposition 3.1. Given k_t, it holds that

$$\frac{d\hat{LS}_t}{d\mu} > 0.$$

Hence, in the short run, a higher life expectancy increases the labor share. To gain intuition for this finding express $\hat{LS}_t$ as

$$\hat{LS}_t = \frac{\hat{\omega}_t h(\hat{\omega}_t)}{\hat{\omega}_t h(\hat{\omega}_t) + \gamma i(\hat{\omega}_t)}$$

and consider the decomposition

$$\frac{d\hat{LS}_t}{d\mu} = \left[\underbrace{\frac{\partial \hat{LS}_t}{\partial \hat{\omega}_t}}_{(+)} + \underbrace{\frac{\partial \hat{LS}_t}{\partial h}}_{(+)} \underbrace{\frac{\partial h(\hat{\omega}_t)}{\partial \hat{\omega}_t}}_{(-)} + \underbrace{\frac{\partial \hat{LS}_t}{\partial i_t}}_{(-)} \underbrace{\frac{\partial i(\hat{\omega}_t)}{\partial \omega_t}}_{(+)}\right] \underbrace{\frac{d\hat{\omega}_t}{d\mu}}_{(-)} > 0.$$

Here, the positive overall effect of μ on $\hat{LS}_t$ obtains even though $(\partial \hat{LS}_t / \partial \hat{\omega}_t)(d\hat{\omega}_t / d\mu) < 0$. This channel is dominated since weaker automation incentives imply more hours per task and lower investment outlays.

4.2 Declining Fertility

A lower fertility rate of cohort t induces the following changes to the equilibrium wage and the labor productivity per hour worked at $t+1$.

Corollary 4.4 *(Short-Run Effects of g_L: Equilibrium Wage and Productivity Growth per Hour Worked)*

Consider the labor market equilibrium of Proposition 3.1 at k_{t+1}. Then, it holds that

$$\frac{d\hat{w}_{t+1}}{dg_L} < 0 \quad \textit{and} \quad \frac{d\hat{q}_{t+1}}{dg_L} < 0.$$

Intuitively, a lower g_L reduces the labor supply at $t+1$ at the extensive margin. This increases the equilibrium wage since the aggregate demand for hours worked is steeper than the aggregate supply.[16] To see this write the labor market equilibrium condition (3.4) for $t+1$ as $H^d(\hat{\omega}_{t+1}) = H^s(\hat{\omega}_{t+1}, L_t(1+g_L), w_c)$. Total differentiation gives

$$\frac{d\hat{\omega}_{t+1}}{dg_L} = \frac{\overbrace{\frac{\partial H^s(\hat{\omega}_{t+1}, L_{t+1}, w_c)}{\partial g_L}}^{(+)} L_t}{\underbrace{\frac{\partial H^d(\hat{\omega}_t)}{\partial \omega_{t+1}} - \frac{\partial H^s(\hat{\omega}_{t+1}, L_{t+1}, w_c)}{\partial \omega_{t+1}}}_{(-)}} < 0,$$

since $\partial H^d(\hat{\omega}_{t+1})/\partial\omega_{t+1} < \partial H^s(\hat{\omega}_{t+1}, L_{t+1}, w_c)/\partial\omega_{t+1} < 0$.

Again, a complementary intuition that emphasizes the role of automation and the elastic supply of hours worked for the effect of changing g_L on $\hat{\omega}_{t+1}$ can be gained from the labor market equilibrium condition at $t+1$ as stated in (3.5). Here, total differentiation delivers

$$\frac{d\hat{\omega}_{t+1}}{dg_L}\frac{1+g_L}{\hat{\omega}_{t+1}} = \left[\frac{\hat{\omega}_{t+1}}{h(\hat{\omega}_{t+1})}\frac{\partial h(\hat{\omega}_{t+1})}{\partial\omega_{t+1}} + \frac{\hat{\omega}_{t+1}}{N(\hat{c}_{t+1})}\frac{\partial N(\hat{c}_{t+1})}{\partial c_{t+1}}\frac{\partial c(\hat{\omega}_{t+1})}{\partial\omega_{t+1}} + \nu\right]^{-1} < 0. \quad (4.2)$$

Hence, the negative sign of $d\hat{\omega}_{t+1}/dg_L$ obtains since the reinforcing impact of a higher $\hat{\omega}_{t+1}$ on the rationalization effect, represented by $\partial h(\hat{\omega}_{t+1})/\partial\omega_{t+1} < 0$, and on the task expansion effect, represented by $(\partial N(\hat{c}_{t+1})/\partial c_{t+1})(\partial c(\hat{\omega}_{t+1})/\partial\omega_{t+1}) < 0$, dominates the effect on the supply of hours worked, represented by ν.

[16] In Figure 3.1 a reduction of the labor supply at the extensive margin corresponds to a smaller L_t. This shifts the aggregate supply of hours worked downwards (not shown). Hence, $\hat{\omega}_t$ increases.

Denote the functional relationship between $\hat{\omega}_{t+1}$ and g_L implied by the labor market equilibrium at $t+1$ by $\hat{\omega}_{t+1} = \hat{\omega}(g_L)$. Then, $\hat{q}_{t+1} = q(\hat{\omega}(g_L))$ and a decline in fertility increases the growth rate of the labor productivity per hour worked at $t+1$ as

$$\frac{d\hat{q}_{t+1}}{dg_L} = \underbrace{\frac{\partial q(\hat{\omega}_{t+1})}{\partial \omega_{t+1}}}_{(+)} \underbrace{\frac{d\hat{\omega}_{t+1}}{dg_L}}_{(-)} < 0.$$

Given k_{t+1}, the equilibrium amount of performed tasks is $\hat{N}_{t+1} = A_t (1 + q(\hat{\omega}_{t+1})) \hat{H}_{t+1}$. Accordingly, short-run GDP in absolute and per-capita terms at $t+1$ is, respectively, $G\hat{D}P_{t+1} = F(K_{t+1}, \hat{N}_{t+1}) - \hat{N}_{t+1} i(\hat{\omega}_{t+1})$ and $g\hat{d}p_{t+1} = G\hat{D}P_{t+1}/P_{t+1}$, where $P_{t+1} = L_t(1 + g_L + \mu)$ is the population at $t+1$. Through the labor market equilibrium, i.e., $\hat{\omega}_{t+1} = \hat{\omega}(g_L)$, $G\hat{D}P_{t+1}$ hinges on g_L.

Corollary 4.5 *(Short-Run Effects of g_L: GDP and gdp)*

Consider the labor market equilibrium of Proposition 3.1 at k_{t+1}. Then, it holds that

$$\frac{dG\hat{D}P_{t+1}}{dg_L} > 0 \quad \textit{and} \quad \frac{dg\hat{d}p_{t+1}}{dg_L} \gtreqless 0.$$

Hence, GDP falls in response to a decline in g_L whereas per-capita GDP may even increase. The intuition is as follows. The higher equilibrium wage induced by a lower fertility rate has two effects on $G\hat{D}P_{t+1}$. First, for a given level of employment, it strengthens the incentive to automate and boosts the growth rate of the labor productivity per hour worked. However, this channel has no first-order effect on $G\hat{D}P_{t+1}$ since $\hat{q}_{t+1}$ is profit-maximizing. Second, for a given growth rate of the labor productivity per hour worked, the level of employment declines. As a consequence, $G\hat{D}P_{t+1}$ falls.

The response of $G\hat{D}P_{t+1}$ to a change in g_L can be expressed as

$$\begin{aligned} \frac{dG\hat{D}P_{t+1}}{dg_L} &= \hat{w}_{t+1} \left(\frac{\partial H^d(\hat{\omega}_{t+1})}{\partial \omega_{t+1}} \frac{d\hat{\omega}_{t+1}}{dg_L} \right), \\ &= \hat{w}_{t+1} H^d(\hat{\omega}_{t+1}) \times \left(\frac{L_t}{L_{t+1}} \right) \times \Psi(\hat{\omega}_{t+1}). \end{aligned} \tag{4.3}$$

The first line shows that the response of $G\hat{D}P_{t+1}$ is the product of the value added of each hour worked, $\hat{w}_{t+1}$, and the change in the equilibrium amount of hours worked. The second line shows that the latter product can be split up into three factors, i.e., the value added per hour worked before the change occurs, $\hat{w}_{t+1} H^d(\hat{\omega}_{t+1})$, the growth factor of the extensive margin of the labor supply induced by the change in g_L, L_t/L_{t+1}, and the elasticity of the equilibrium level of hours worked with respect to g_L, $\Psi(\hat{\omega}_{t+1})$. This elasticity exceeds unity since $\nu > 0$ (see the proof of Corollary 4.5).

With (4.3) the condition under which a decline in g_L increases $g\hat{d}p_{t+1}$ can be expressed as

$$\frac{dg\hat{d}p_{t+1}}{dg_L} < 0 \qquad \Leftrightarrow \qquad LS\left(\hat{\omega}_{t+1}\right) \times \left(\frac{L_t}{L_{t+1}}\right) \times \Psi\left(\hat{\omega}_{t+1}\right) < \frac{1}{\mu + 1 + g_L}. \tag{4.4}$$

The left-hand side is the growth rate of $G\hat{D}P_{t+1}$ induced by a change in g_L. It is equal to the product of the equilibrium labor share before the change in g_L occurs, $LS\left(\hat{\omega}_{t+1}\right) = \hat{w}_{t+1}H^d\left(\hat{\omega}_{t+1}\right)/G\hat{D}P_{t+1}$ and $\left(L_t/L_{t+1}\right) \times \Psi\left(\hat{\omega}_{t+1}\right)$. The right-hand side is the population growth rate induced by a change in g_L. Hence, a decline in fertility implies an increase in per-capita GDP if the induced growth rate of GDP falls short of the induced population growth rate. This condition is easier to satisfy the lower μ and the higher g_L.[17]

Denote the labor share evaluated at the labor-market equilibrium at $t+1$ by $\hat{LS}_{t+1} = LS\left(\hat{\omega}_{t+1}\right)$ where $\hat{\omega}_{t+1} = \hat{\omega}\left(g_L\right)$. Corollary 2.3 and 4.4 imply that $\hat{LS}_{t+1}$ falls in response to a lower fertility rate since stronger incentives to automate reduce the equilibrium labor share.

Corollary 4.6 *(Short-Run Effects of g_L: Labor Share)*

Consider the labor market equilibrium of Proposition 3.1 at k_{t+1}. Then, it holds that

$$\frac{d\hat{LS}_{t+1}}{dg_L} > 0.$$

To gain intuition for this finding express $\hat{LS}_{t+1}$ as

$$\hat{LS}_{t+1} = \frac{\hat{\omega}_{t+1}h\left(\hat{\omega}_{t+1}\right)}{\hat{\omega}_{t+1}h\left(\hat{\omega}_{t+1}\right) + \gamma i\left(\hat{\omega}_{t+1}\right)}$$

and consider the decomposition

$$\frac{d\hat{LS}_{t+1}}{dg_L} = \left[\underbrace{\frac{\partial \hat{LS}_{t+1}}{\partial \hat{\omega}_{t+1}}}_{(+)} + \underbrace{\frac{\partial \hat{LS}_{t+1}}{\partial h}}_{(+)}\underbrace{\frac{\partial h\left(\hat{\omega}_{t+1}\right)}{\partial \hat{\omega}_{t+1}}}_{(-)} + \underbrace{\frac{\partial \hat{LS}_{t+1}}{\partial i_{t+1}}}_{(-)}\underbrace{\frac{\partial i\left(\hat{\omega}_{t+1}\right)}{\partial \omega_{t+1}}}_{(+)}\right]\underbrace{\frac{d\hat{\omega}_{t+1}}{dg_L}}_{(-)} > 0.$$

Hence, a lower g_L reduces $\hat{LS}_{t+1}$ even though $\left(\partial \hat{LS}_{t+1}/\partial \hat{\omega}_{t+1}\right)\left(d\hat{\omega}_{t+1}/dg_L\right) < 0$. The latter channel is dominated since stronger automation incentives imply fewer hours per task and higher investment outlays.

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[17]Consider a period length of 30 years and suppose that the annual productivity growth rate per hour worked is 2%. Then, $\hat{\omega}_{t+1} = (1.811)^2\alpha$ and $\hat{q}_{t+1} = 0.811$. Moreover, let $\gamma = 1/3$, $\nu = 1/4$, and $g_L = 0.35$, which corresponds to an annual fertility rate of 1%. Then, condition (4.4) is satisfied for $\mu < .75$. If instead $g_L = 0.56$, which corresponds to an annual fertility rate of 1.5%, then condition (4.4) is satisfied for $\mu < 0.87$. For many industrialized countries the range of μ interpreted, e. g., as the survival probability for males to age 65 between 1960-2017, is $[0.5, 0.9]$ (see Appendix B.1). Hence, condition (4.4) is sensitive to country-specific parameters.

5 Long-Run Macroeconomic Implications of Population Aging

This section derives the long-run, i. e., *steady-state*, implications of a permanent increase in life expectancy and a permanent decline in the fertility rate for automation, growth, and factor shares. It proves useful to start the analysis with the structural properties of the steady state.[18]

5.1 Structural Properties of the Steady State

The steady-state evolution of all endogenous variables is as follows.

Proposition 5.1 *(Structural Properties of the Steady State)*

Consider the steady state of Proposition 3.2. Then, aggregate technological knowledge grows at rate $q^* > 0$. Moreover,

a) $$\frac{a_{t+1}}{a_t} = 1 + q^*, \quad \frac{h_{t+1}}{h_t} = \frac{1}{1+q^*}, \quad i_t = i^* > 0, \quad c_t = c^*,$$

b) $$\frac{w_{t+1}}{w_t} = \frac{\hat{w}_{t+1}}{\hat{w}_t} = 1 + q^*, \quad R_t = R^* > 0,$$

c) $$\frac{h^s_{t+1}}{h^s_t} = \frac{1}{(1+q^*)^\nu}, \quad \frac{c^y_{t+1}}{c^y_t} = \frac{c^o_{t+1}}{c^o_t} = \frac{s_{t+1}}{s_t} = (1+q^*)^{1-\nu},$$

d) $$\frac{\hat{H}_{t+1}}{\hat{H}_t} = (1+q^*)^{-\nu}(1+g_L), \quad \frac{Y_{t+1}}{Y_t} = \frac{K_{t+1}}{K_t} = \frac{I_{t+1}}{I_t} = \frac{N_{t+1}}{N_t} = (1+q^*)^{1-\nu}(1+g_L).$$

The intuition is as follows. Since $\omega^* > \alpha$ firms undertake automation investments that support a strictly positive growth rate of aggregate technological knowledge, $q^* > 0$. On the production side, this means that the labor productivity per hour worked increases at this rate. Accordingly, there will be rationalization at the level of each task, i. e., $h_{t+1}/h_t < 1$. Automation investments per task remain constant over time. The real wage inherits the growth rate of aggregate technological knowledge since by definition $\hat{w}_t = A_{t-1}\hat{\omega}^*$. As wages and the productivity per working hour grow at the same rate and $i_t = i^*$ the costs per task are time-invariant, i. e., $c_t = c^*$.

On the household side, wage growth implies a declining individual supply of hours worked. The key implication is that wage income, $w_t h^s_t$, grows at a factor $(1+q^*)^{1-\nu}$,

[18] The calibration exercise presented in Section B.3 of the Online Appendix reveals that the quantitative properties of the steady state are broadly consistent with the long-run evolution of industrialized economies over the last century.

which is also the growth factor of consumption in both periods of life and of individual savings.[19]

At the level of economic aggregates, the evolution of the equilibrium amount of hours worked reflects a decline at the intensive margin, $(1+q^*)^{-\nu}$, and an expansion at the extensive margin, $1+g_L$. From the accumulation equation (3.6) it is obvious that fixed capital grows with a factor $(1+q^*)^{1-\nu}(1+g_L)$. Total output, Y_t, the aggregate demand for automation investments, I_t, the number of tasks, N_t, and, hence, GDP_t inherit this trend.

Finally, observe that the steady state is a balanced growth path as the labor share, the ratios K_t/Y_t, I_t/Y_t, and $\left(\mu L_{t-1}c_t^o + L_t c_t^y\right)/Y_t$ as well as the real rental rate of capital remain constant over time.

5.2 Increasing Longevity, Declining Fertility, and the Long Run

The following corollary shows that of two otherwise identical economies the one with a higher life expectancy and/or a lower fertility rate enjoys faster steady-state growth of labor productivity per hour worked. As a higher μ and a lower g_L increase the steady-state OADR it follows for the long run that per-capita variables grow faster in the older economy.

Corollary 5.1 *(Long-Run Effects: Productivity Growth per Hour Worked)*

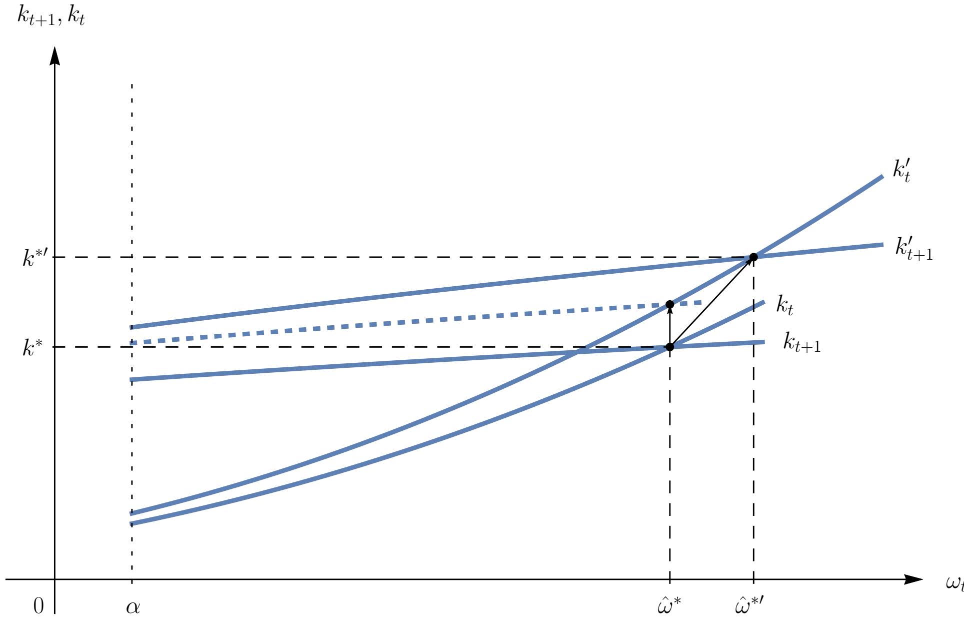
Consider the steady state of Proposition 3.2. It holds that

$$\frac{dq^*}{d\mu} > 0 \quad \textit{and} \quad \frac{dq^*}{dg_L} < 0.$$

The effect of a permanent increase in μ on q^* reflects three channels. These are illustrated in Figure 5.1 which builds on Figure 3.2. Initially the economy has a survival probability equal to μ and starts in the steady state (ω^*, k^*). The new steady state corresponding to $\mu' > \mu$ is $(\omega^{*\prime}, k^{*\prime})$. The first channel is the short-run effect identified in Corollary 4.1 and propagates through the labor market. A higher μ increases the individual and the aggregate supply of hours worked. Accordingly, the labor-market equilibrium locus shifts upwards (see the curve denoted by k_t'), and, given k^*, the equilibrium wage, falls. The second and the third channel operate through the capital market. Here, a greater μ increases savings for two reasons. First, the wage income increases with the individual supply

[19]Observe that a higher ν increases the savings rate. Therefore, q^* also increases. The effect of a higher ν on the growth factor of per-capita variables, $(1+q^*)^{1-\nu}$, has to take the individual labor supply decision into account. Analytically, one can show that a higher ν increases this growth factor if q^* is sufficiently small. This finding remains true for the parameter constellation considered in Footnote 17.

Figure 5.1: The Effect of a Higher Life Expectancy on the Steady State.



Note: In response to a permanent increase in life expectancy from μ to μ' the steady state of the economy switches from (ω^*, k^*) to $(\omega^{*\prime}, k^{*\prime})$. The labor market equilibrium condition (3.5) denoted, respectively, by k_t and k'_t shifts upwards. The capital market equilibrium locus of (3.7) denoted, respectively, by k_{t+1} and k'_{t+1} shifts upwards for two reasons. First, individual wage income increases as individuals work more hours, and, second, the propensity to save increases. The dashed blue line shows the upward shift of the capital market equilibrium locus that reflects only the increase in the supply of hours worked. This shift leaves $\hat{\omega}^*$ unchanged.

of hours worked (second channel, see Corollary 2.4). Given ω_t, this shifts the capital market equilibrium locus in Figure 5.1 upwards (see the dashed blue line). Second, the individual propensity to save increases (third channel, see Corollary 2.4). In Figure 5.1, this effect shifts the capital market equilibrium locus even further upwards (see the curve denoted by k'_{t+1}). As a result, the new steady state has $k^{*\prime} > k^*$, $\hat{\omega}^{*\prime} > \hat{\omega}^*$, and, $q^{*\prime} > q^*$.[20] Hence, population aging through increased longevity induces faster steady-state growth of per-capita variables.

A permanent decline in the fertility rate, g_L, means higher savings per unit of next period's workers, i. e., Ω increases. This shifts the capital market equilibrium locus in Figure 5.1 upwards (not shown). As a consequence, the steady state corresponding to $g'_L < g_L$ has $k^{*\prime} > k^*$, $\hat{\omega}^{*\prime} > \hat{\omega}^*$, and, $q^{*\prime} > q^*$. Hence, population aging through a permanent decline in fertility also leads to faster long-run growth of per-capita variables.

Population aging also affects the long-run growth rate of GDP and gdp that are, respectively, given by $g^*_{GDP} = (1+q^*)^{1-\nu}(1+g_L) - 1$ and $g^*_{gdp} = (1+q^*)^{1-\nu} - 1$.

[20]The proof of Corollary 5.1 reveals that, as shown in Figure 5.1, the third channel is responsible for the increase in $\hat{\omega}^*$.

Corollary 5.2 *(Long-Run Effects: Growth of GDP and gdp)*

Consider the steady state of Proposition 3.2. It holds that

$$\frac{dg^*_{GDP}}{d\mu} > 0, \quad \frac{dg^*_{GDP}}{dg_L} > 0, \quad \frac{dg^*_{gdp}}{d\mu} > 0, \quad \frac{dg^*_{gdp}}{dg_L} < 0.$$

Hence, irrespective of its source population aging increases the long run growth rate of gdp since it speeds up the growth rate of labor productivity per hour worked. If a higher longevity is the source of population aging then this force makes GDP grow faster, too. However, if population aging is due to a decline in fertility then the growth rate of GDP falls since the decline in the growth rate of the work force dominates.

Finally, let $\hat{LS}^* = LS(\omega^*)$ denote the steady-state labor share.

Corollary 5.3 *(Long-Run Effects: Labor Share)*

Consider the steady state of Proposition 3.2. It holds that

$$\frac{d\hat{LS}^*}{d\mu} < 0 \quad \text{and} \quad \frac{d\hat{LS}^*}{dg_L} > 0.$$

Hence, population aging reduces the steady-state labor share irrespective of its source. From Corollaries 4.3, 4.6, and 5.1, the intuition is that a higher μ or a lower g_L increases q^* which reduces $\hat{LS}^*$.

6 Concluding Remarks

Since the 1960ies population aging has changed the macroeconomic environment that firms operate in. An increasing longevity and a declining fertility have been and will remain the key drivers of this process in many industrialized countries. At the same time, the scale and scope of automation substantially widened through technological progress. This paper shows that population aging implies behavioral adjustments that affect the incentives to automate, hence, economic growth, and factor shares. Table 6.1 summarizes the results of the analysis.

The comparative static effects of an increase in μ and of a decline in g_L capture the key drivers of population aging. Both changes lift the old-age dependency ratio as stated in (2.16). However, an increase in this ratio may still occur if, e. g., both longevity and fertility increase simultaneously and the growth in the former dominates the growth of the

Table 6.1: The Effects of Changing μ and g_L for the Short and the Long Run.

Short-Run Effects

	$\hat{q}$	$\hat{GDP}$	$\hat{gdp}$	$\hat{LS}$
μ	$(-)$	$(+)$	$(+)$	$(+)$
g_L	$(-)$	$(+)$	$(+/-)$	$(+)$

Long-Run Effects

	q^*	g^*_{GDP}	g^*_{gdp}	LS^*
μ	$(+)$	$(+)$	$(+)$	$(-)$
g_L	$(-)$	$(+)$	$(-)$	$(+)$

Note: The short-run effects of a change in μ at t apply to period-t variables, a change in g_L at t applies to period-$t+1$ variables.

latter. Then, one may use the analytical results derived in Section 4 and 5 in a quantitative analysis to gauge the overall effect of population aging on automation, growth, and factor shares.

To the extent that economic growth is due to automation (and the accumulation of fixed capital), the comparative statics in Table 6.1 suggest conjectures about the effects of population aging on economic growth and the labor share that subsequent empirical research may want to address. For instance, in the long run older economies are predicted to grow faster and to have a lower labor share. However, Table 6.1 also emphasizes that the identification of these two effects needs to distinguish between the sources of aging as well as the long and the short run.

The present paper gives rise to several new questions that a comprehensive understanding of the effects of population aging on automation, growth, and factor shares needs to address. One concerns the role of perfect competition and of the knowledge accumulation process. For instance, one may wonder whether the short-run effects of population aging through automation remain of second order when firms produce differentiated goods and/or when they internalize the effect of their automation investments on the evolution of technological knowledge.

Another concerns the role of increasing educational attainments that have been observed since the 1960ies (Barro and Lee (2018)). Intuition suggests that the expectation of a longer working life may increase the rate of return of an educational investment. At the same time, new automation technologies may depreciate the acquired human capital so that the potential effect of these tendencies on the incentive to automate, growth, and factor shares remains elusive.

Finally, one may want to allow for alternative ways to expand the supply of hours worked in response to aging. They include an endogenous retirement age for individuals or an extensive margin of the labor supply for households. The results of the present paper suggest a tendency to retire later or to expand the extensive margin in response to a higher life expectancy. Both are in line with the empirical evidence (see, e. g., Bloom, Canning, Mansfield, and Moore (2007), Aísa, Pueyo, and Sanso (2012)). As these adjustments induce a positive level effect on the aggregate labor supply, one may conjecture that at least the qualitative effects for the short run are similar to those derived in the analysis above. I leave the detailed analysis of these issues for future research.

A Appendix: Proofs

The proofs of Proposition 2.2, 2.3, as well as of Corollary 2.2, 2.3, 4.6, and 5.3 are given in the main text.

A.1 Proof of Proposition 2.1

Given $q(\omega_t)$ of (2.12), equation (2.8) delivers $h_t = 1/\left(A_{t-1}\left(1+q\left(\omega_t\right)\right)\right) \equiv h\left(\omega_t\right)/A_{t-1}$. From (2.4) $i_t = i\left(\omega_t\right)$. Since the wage cost per task is $w_t h_t = \omega_t h\left(\omega_t\right)$, we have $c_t = \omega_t h\left(\omega_t\right) + i\left(\omega_t\right) \equiv c\left(\omega_t\right)$. Continuity of these functions follows since $\lim_{\omega_t \downarrow \alpha} q\left(\omega_t\right) = 0$. The remaining arguments that complete the proof are straightforward or given in the main text. ■

A.2 Proof of Corollary 2.1

If $\omega_t > \alpha$ then $q_t > 0$ and the rationalization effect follows since $\left(A_{t-1}\left(1+q_t\right)\right)^{-1} < A_{t-1}^{-1}$. The productivity effect follows since c_t is the solution to (2.10) and $c\left(\omega_t\right)|_{\omega_t=\alpha} = \omega_t$. ■

A.3 Proof of Proposition 2.4

For ease of notation I shall most often suppress the time argument. Consider problem (2.20). Since preferences are increasing in c^o both periodic budget constraints will hold as equalities and can be merged. Accordingly, the Lagrangian of this problem is

$$\mathcal{L} = \ln c^y + \ln\left(1 - \phi\left(1-l\right)\left(c^y\right)^{\frac{\nu}{1-\nu}}\right) + \mu\beta \ln c^o + \lambda\left[w\left(1-l\right) - c^y - \frac{\mu c^o}{R}\right]. \tag{A.1}$$

Corner solutions involving $c^y = c^o = 0$ and $l = 1$ can be excluded since U satisfies the Inada conditions and $l = 1$ implies no income. Hence, with $x \equiv \left(1-l\right)\left(c^y\right)^{\frac{\nu}{1-\nu}}$ the respective first-order Kuhn-Tucker conditions read as follows:

$$\frac{\partial \mathcal{L}}{\partial c^y} = \frac{1-\nu-\phi x}{c^y(1-\nu)(1-\phi x)} - \lambda = 0, \tag{A.2}$$

$$\frac{\partial \mathcal{L}}{\partial l} = \frac{\phi\left(c^y\right)^{\frac{\nu}{1-\nu}}}{1-\phi x} - \lambda w \leq 0, \quad \text{with strict inequality if } l_t = 0, \tag{A.3}$$

$$\frac{\partial \mathcal{L}}{\partial c^o} = \frac{\beta}{c^o} - \frac{\lambda}{R} = 0, \tag{A.4}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = w\left(1-l\right) - c^y - \frac{\mu c^o}{R} = 0. \tag{A.5}$$

Suppose $l > 0$. Then, upon multiplication by $(1-l)$, condition (A.3) may be written as

$$\frac{\phi x}{\left(1-\phi x\right) w\left(1-l\right)} = \lambda. \tag{A.6}$$

Using the latter to replace λ in (A.2) and (A.4) delivers, respectively,

$$c^y = \left(\frac{1}{\phi x} - \frac{1}{1-\nu}\right) w(1-l) \tag{A.7}$$

and

$$c^o = \beta R\left(\frac{1}{\phi x} - 1\right) w(1-l). \tag{A.8}$$

With (A.7) and (A.8) in the budget constraint (A.5) I obtain

$$\phi x_t = \phi x = \frac{(1+\mu\beta)(1-\nu)}{1+(1+\mu\beta)(1-\nu)} \in (0,1). \tag{A.9}$$

Using (A.9) in (A.7), (A.8), and (2.19) delivers (2.21). Since the optimal plan satisfies Assumption 1 I have $1 > \nu(1+\mu\beta)$, hence, $c^y > 0$.

From the definition of x with $h^s = 1 - l$ it holds that $c^y = \left(x\left(h^s\right)^{-1}\right)^{\frac{1-\nu}{\nu}}$. Replacing c^y with this expression in (2.21) and solving for h^s delivers h^s_t. Using the latter in (2.21) delivers c_t and s_t. Then, c^o_{t+1} is obtained from the budget when old. Clearly, $h^s_t \leq 1$ as long as $w_t \geq w_c$. In accordance with this, $w < w_c$ implies a strict inequality in (A.3).

To see that the solution identified by the Lagrangian (A.1) is indeed a global maximum if $\nu < \bar{\nu}(\mu\beta)$ consider first the leading principal minors of the Hessian matrix of $U(c^y, l, c^o)$, i. e.,

$$\begin{aligned}
D_1\left(c^y, l, c^o\right) &= -\frac{(1-\nu-\phi x)^2 + \nu\phi x(1-\phi x)}{\left(c^y(1-\nu)(1-\phi x)\right)^2}, \\
D_2\left(c^y, l, c^o\right) &= \frac{\phi^2\left(1-2\nu-(1-\nu)\phi x\right)}{\left(c^y\right)^{\frac{2(1-2\nu)}{1-\nu}}(1-\nu)^2(1-\phi x)^3}, \\
D_3\left(c^y_t, l_t, c^o_{t+1}\right) &= -\frac{\mu\beta}{\left(c^o\right)^2} D_2\left(c^y_t, l_t, c^o_{t+1}\right).
\end{aligned}$$

First, we have $-D_1(c^y, l, c^o) > 0$. Second, observe that $D_2(c^y, l, c^o) > 0$ and $-D_3(c^y, l, c^o) > 0$ hold if and only if condition (2.18) holds. Hence, U is strictly concave for all $(c^y, l, c^o) \in \mathcal{P}$.

What remains to be shown is that the solution identified by the Lagrangian satisfies condition (2.18). With ϕx of (A.9) this is the case if and only if

$$\frac{1-2\nu}{1-\nu} > \frac{(1+\mu\beta)(1-\nu)}{1+(1+\mu\beta)(1-\nu)}$$

or

$$\nu^2(1+\mu\beta) - \nu(3+\mu\beta) + 1 > 0.$$

It is not difficult to show that the latter condition is satisfied if and only if $\nu < \bar{\nu}(\mu\beta)$ as stated in Assumption 1.

Finally, observe that surviving members of cohort 0 satisfy their budget constraint when old as equality, i. e., we have $c^o_1 = R_1 s_0/\mu > 0$. ■

A.4 Proof of Corollary 2.4

Some algebra reveals that

$$\frac{\partial w_c}{\partial \mu} = \frac{\beta w_c}{\nu(1+\mu\beta)(1+(1-\nu)(1+\mu\beta))(1-\nu(1+\mu\beta)} > 0.$$

It follows that $\partial h^s_t/\partial\mu > 0$. From the definition of w_c and Proposition 2.4, c^y_t may be written as

$$c^y_t = \left(\frac{1-\nu(1+\mu\beta)}{\phi\left(1+(1+\mu\beta)(1-\nu)\right)}\right)^{1-\nu} w_t^{1-\nu}.$$

Hence,

$$\frac{\partial c^y_t}{\partial \mu} = \frac{-(1-\nu)\beta w_t^{1-\nu}}{\phi^{1-\nu}(1+(1-\nu)(1+\mu\beta))^{2-\nu}(1-\nu(1+\mu\beta))^{\nu}} < 0.$$

The sign of $\partial s_t/\partial\mu > 0$ follows since the marginal propensity to save in (2.21) increases in μ and $\partial h_t^s/\partial\mu > 0$. Finally, using s_t in the budget constraint of a surviving old delivers

$$c_{t+1}^o = \frac{\beta R_{t+1} w_t^{1-\nu}}{\phi^{1-\nu}(1+(1-\nu)(1+\mu))^{1-\nu}(1-\nu(1+\mu))^\nu}.$$

Hence, by Assumption 1

$$\frac{\partial c_{t+1}^o}{\partial\mu} = -\frac{\left(\nu^2(1+\mu\beta)-\nu(3+\mu\beta)+1\right)\beta^2 R_{t+1} w_t^{1-\nu}}{\phi^{1-\nu}(1+(1+\nu)(1+\mu\beta))^{2-\nu}(1-\nu(1+\mu\beta))^{1+\nu}} < 0.$$

■

A.5 Proof of Proposition 3.1

Under Assumption 2, H_t^d is given by (3.2) whereas H_t^s is given by (3.3). Hence, (3.4) delivers (3.5). Denote the right-hand side of (3.5) by $RHS(\omega_t)$ where $RHS : [\alpha,\infty) \to [\underline{k}_c,\infty)$. Then, $RHS(\alpha) = \underline{k}_c > 0$. Moreover, since $\nu < 1/2$ we have $RHS'(\omega_t) > 0$ for $\omega_t > \alpha$ and $\lim_{\omega_t\to\infty} RHS(\omega_t) = \infty$. Hence, for equation (3.5) to be satisfied for any value $\omega_t > \alpha$ it is necessary and sufficient to have $RHS(\alpha) < k_t$ or $k_t > \underline{k}_c$. Then, the above-mentioned properties of $RHS(\omega_t)$ assure that there is indeed a unique $\hat{\omega}_t > \alpha$ that satisfies (3.5). By the implicit function theorem, the function $\omega(k_t)$ has the indicated properties. ■

A.6 Proof of Proposition 3.2

First, observe that k_t is a state variable of the inter-temporal general equilibrium. Indeed, given $k_t > \underline{k}_c$, the labor market determines $\hat{\omega}_t = A_{t-1}\hat{w}_t > \alpha$. Hence, Proposition 2.1 delivers q_t, a_t, i_t, and c_t. Proposition 2.2 and (3.2) determine N_t, Y_t, I_t, and H_t^d. Hence, (2.13) delivers R_t. On the household side, Proposition 2.4 gives h_t^s, l_t, c_t^y, c_t^o, s_t. Finally, K_{t+1} follows from (3.6).

Second, consider (3.5) and replace $\hat{\omega}_t$ by $\omega_t = (k_{t+1}/\Omega)^{1/(1-\nu)}$ from (3.7). Then, the equilibrium difference equation (3.8) is given by

$$k_t = \frac{\underline{k}_c}{\alpha^{\frac{1}{2}-\nu}}\left(\frac{k_{t+1}}{\Omega}\right)^{\frac{1-2\nu}{2(1-\nu)}}\left(\frac{2}{\sqrt{\alpha}}\left(\frac{k_{t+1}}{\Omega}\right)^{\frac{1}{2(1-\nu)}}-1\right)^{\frac{1}{\gamma}}. \tag{A.10}$$

Denote the right-hand side of (A.10) by $RHS(k)$. The latter satisfies $\lim_{k\downarrow\bar{k}_c} RHS(k) = \underline{k}_c$, is continuous, and, since $\nu < 1/2$, increasing with $\lim_{k\to\infty} RHS(k) = \infty$. Hence, (A.10) assigns to each $k_t > \underline{k}_c$ a unique $k_{t+1} > \bar{k}_c$. To see that there is a unique fixed point $k = RHS(k)$ write (A.10) for $k_t = k_{t+1} = k$ as

$$k = Z_1 k^{\frac{1-2\nu}{2(1-\nu)}}\left(Z_2 k^{\frac{1}{2(1-\nu)}}-1\right)^{\frac{1}{\gamma}}$$

or

$$k^{\frac{\gamma}{2(1-\nu)}} = Z_1^\gamma Z_2 k^{\frac{1}{2(1-\nu)}} - Z_1^\gamma, \tag{A.11}$$

where

$$Z_1 \equiv \frac{\underline{k}_c}{\alpha^{\frac{1}{2}-\nu}}\left(\frac{1}{\Omega}\right)^{\frac{1-2\nu}{2(1-\nu)}} > 0 \quad\text{and}\quad Z_2 \equiv \frac{2}{\sqrt{\alpha}}\left(\frac{1}{\Omega}\right)^{\frac{1}{2(1-\nu)}} > 0.$$

Since $0 < \gamma < 1$ and $0 < \nu < 1/2$ it holds that $0 < \gamma/(2(1-\nu)) < 1/(2(1-\nu)) < 1$. Hence, the left and the right-hand side of (A.11) are concave with a unique intersection at some $k > 0$. Moreover, a straightforward graphical argument in (k_{t+1}, k_t) - space reveals that $k > \bar{k}_c$ and that k is stable for all $k_1 > \underline{k}_c$ (see, e. g., Galor (2007)). $A_0 > w_c/\alpha$ ensures Assumption 2. ■

A.7 Proof of Corollary 4.1

Since the right-hand side of (3.5) is increasing in $\hat{\omega}_t$ and $\partial w_c/\partial\mu > 0$ there is by the implicit function theorem a function $\hat{\omega}_t = \hat{\omega}(\mu)$ with $d\hat{\omega}_t/d\mu < 0$. Hence, $d\hat{w}_t/d\mu = (\partial\hat{w}_t/\partial\omega_t)(d\hat{\omega}_t/d\mu) < 0$. The sign of $d\hat{q}_t/d\mu$ follows with Proposition 2.1. ■

A.8 Proof of Corollary 4.2

Let $\hat{H}_t = H^d(\hat{\omega}_t) = H^d(\hat{\omega}(\mu))$ where $\hat{\omega}_t = \hat{\omega}(\mu)$ is defined in the Proof of Corollary 4.1. With $\hat{N}_t = A_{t-1}(1+q(\hat{\omega}(\mu)))H^d(\hat{\omega}(\mu))$ define $G\hat{D}P_t = F(K_t, \hat{N}_t) - \hat{N}_t i(\hat{\omega}(\mu)) \equiv GDP(\hat{\omega}(\mu))$. Then, it holds that $dG\hat{D}P_t/d\mu = (\partial GDP(\hat{\omega}_t)/\partial\omega_t)(d\hat{\omega}_t/d\mu)$ where Corollary 4.1 delivers $d\hat{\omega}_t/d\mu < 0$. Some manipulations reveal that

$$\begin{aligned}\frac{\partial GDP(\hat{\omega}_t)}{\partial\omega_t} &= A_{t-1}H^d(\hat{\omega}_t)\frac{\partial q(\hat{\omega}_t)}{\partial\omega_t}\left[F_2 - (1+q(\hat{\omega}_t))\frac{\partial i(\hat{\omega}_t)}{\partial q_t} - i(\hat{\omega}_t)\right] \\ &+ A_{t-1}(1+q(\hat{\omega}_t))\frac{\partial H^d(\hat{\omega}_t)}{\partial\omega_t}\left[F_2 - i(\hat{\omega}_t)\right],\end{aligned} \tag{A.12}$$

where F is evaluated at $(K_t, \hat{N}_t)$. In equilibrium, the bracketed expression in the first line vanishes. To see this, observe that the cost per task in a symmetric configuration is $w_t h_t + i(q_t) = \omega_t/(1+q_t) + i(q_t)$. Minimizing this expression with respect to q_t gives the first-order condition $\omega_t/(1+q_t) = (1+q_t)\partial i(q_t)/\partial q_t$. Hence, the minimized cost per task can be written as $c_t = (1+q_t)\partial i(q_t)/\partial q_t + i(q_t)$. Profit maximization requires conditions (2.13) to hold. Hence, $F_2 = c_t = (1+q_t)\partial i(q_t)/\partial q_t + i(q_t)$ holds in equilibrium. As $\partial H^d(\hat{\omega}_t)/\partial\omega_t < 0$ and $F_2 - i(\hat{\omega}_t) > 0$ it follows that $\partial GDP(\hat{\omega}_t)/\partial\omega_t < 0$, hence, $dG\hat{D}P_t/d\mu > 0$. ■

A.9 Proof of Corollary 4.3

Corollary 2.3 proves $\partial LS_t/\partial\omega_t < 0$. Hence, $d\hat{LS}_t/d\mu = (\partial LS_t/\partial\omega_t)(d\hat{\omega}_t/d\mu) > 0$. ■

A.10 Proof of Corollary 4.4

Consider (3.5) with $k_{t+1} = K_{t+1}/\left(A_t^{1-\nu}L_t(1+g_L)\right)$. Since the right-hand side of (3.5) is increasing in $\hat{\omega}_{t+1}$ there is by the implicit function theorem a function $\hat{\omega}_{t+1} = \hat{\omega}(g_L)$ with $d\hat{\omega}_{t+1}/dg_L < 0$. Hence, $d\hat{w}_{t+1}/dg_L = (\partial\hat{w}_{t+1}/\partial\omega_{t+1})(d\hat{\omega}_{t+1}/dg_L) < 0$. The sign of $d\hat{q}_{t+1}/dg_L$ follows with Proposition 2.1. ■

A.11 Proof of Corollary 4.5

Let $\hat{H}_{t+1} = H^d(\hat{\omega}_{t+1}) = H^d(\hat{\omega}(g_L))$ where $\hat{\omega}_{t+1} = \hat{\omega}(g_L)$ is defined in the proof of Corollary 4.4. With $\hat{N}_{t+1} = A_t(1+q(\hat{\omega}(g_L)))H^d(\hat{\omega}(g_L))$ let $G\hat{D}P_{t+1} = F(K_{t+1}, \hat{N}_{t+1}) - \hat{N}_{t+1}i(\hat{\omega}(g_L)) \equiv GDP(\hat{\omega}(g_L))$. Then, it holds that $dG\hat{D}P_{t+1}/dg_L = (\partial GDP(\hat{\omega}_{t+1})/\partial\omega_{t+1})(d\hat{\omega}_{t+1}/dg_L)$. Some manipulations reveal that

$$\begin{aligned}\frac{\partial GDP(\hat{\omega}_{t+1})}{\partial\omega_{t+1}} &= A_t H^d(\hat{\omega}_{t+1})\frac{\partial q(\hat{\omega}_{t+1})}{\partial\omega_{t+1}}\left[F_2 - (1+q(\hat{\omega}_{t+1}))\frac{\partial i(\hat{\omega}_{t+1})}{\partial q_{t+1}} - i(\hat{\omega}_{t+1})\right] \\ &+ A_t(1+q(\hat{\omega}_{t+1}))\frac{\partial H^d(\hat{\omega}_{t+1})}{\partial\omega_{t+1}}\left[F_2 - i(\hat{\omega}_{t+1})\right],\end{aligned} \tag{A.13}$$

where F is evaluated at $\left(K_{t+1}, \hat{N}_{t+1}\right)$. For the same reason as in the proof of Corollary 4.2, the bracketed expression in the first line vanishes. Hence, as $\partial H^d\left(\hat{\omega}_{t+1}\right)/\partial\omega_{t+1} < 0$ and $F_2 - i\left(\hat{\omega}_{t+1}\right) > 0$ it follows that $\partial GDP\left(\hat{\omega}_{t+1}\right)/\partial\omega_{t+1} < 0$, hence, $dG\hat{D}P_t/dg_L > 0$.

Turning to the effect of g_L on $g\hat{d}p_{t+1}$, observe that $g\hat{d}p_{t+1} \equiv G\hat{D}P_{t+1}/\left(L_t\left(\mu + 1 + g_L\right)\right)$. Hence,

$$\frac{dg\hat{d}p_{t+1}}{dg_L} \gtreqless 0 \quad \Leftrightarrow \quad \frac{dG\hat{D}P_{t+1}}{dg_L} \gtreqless \frac{G\hat{D}P_{t+1}}{\mu + 1 + g_L}.$$

With (A.13) we have

$$\begin{aligned} \frac{dG\hat{D}P_{t+1}}{dg_L} &= A_t\left(1 + q\left(\hat{\omega}_{t+1}\right)\right)\left[F_2 - i\left(\hat{\omega}_{t+1}\right)\right]\frac{\partial H^d\left(\hat{\omega}_{t+1}\right)}{\partial\omega_{t+1}}\frac{d\hat{\omega}_{t+1}}{dg_L}, \\ &= \hat{w}_{t+1}\left(\frac{\partial H^d\left(\hat{\omega}_{t+1}\right)}{\partial\omega_{t+1}}\frac{d\hat{\omega}_{t+1}}{dg_L}\right), \end{aligned} \tag{A.14}$$

where the second line uses the first-order condition (2.13) for N_{t+1}, i. e., $F_2(K_{t+1}, N_{t+1}) = c_{t+1} = w_{t+1}h_{t+1} + i_{t+1}$. The latter implies $w_{t+1} = \left[F_2(K_{t+1}, N_{t+1}) - i_{t+1}\right]/h_{t+1} = A_t\left(1 + q_{t+1}\right)\left[F_2(K_{t+1}, N_{t+1}) - i_{t+1}\right]$.

With (3.5) and (4.2) one finds

$$\begin{aligned} \frac{\partial H^d\left(\hat{\omega}_{t+1}\right)}{\partial\omega_{t+1}}\frac{d\hat{\omega}_{t+1}}{dg_L} &= \frac{H^d\left(\hat{\omega}_{t+1}\right)}{1 + g_L}\left[\frac{\frac{\hat{\omega}_{t+1}}{h(\hat{\omega}_{t+1})}\frac{\partial h(\hat{\omega}_{t+1})}{\partial\omega_{t+1}} + \frac{\hat{\omega}_{t+1}}{N(\hat{c}_{t+1})}\frac{\partial N(\hat{c}_{t+1})}{\partial c_{t+1}}\frac{\partial c(\hat{\omega}_{t+1})}{\partial\omega_{t+1}}}{\frac{\hat{\omega}_{t+1}}{h(\hat{\omega}_{t+1})}\frac{\partial h(\hat{\omega}_{t+1})}{\partial\omega_{t+1}} + \frac{\hat{\omega}_{t+1}}{N(\hat{c}_{t+1})}\frac{\partial N(\hat{c}_{t+1})}{\partial c_{t+1}}\frac{\partial c(\hat{\omega}_{t+1})}{\partial\omega_{t+1}} + \nu}\right] \\ &= \frac{H^d\left(\hat{\omega}_{t+1}\right)}{1 + g_L} \times \Psi\left(\hat{\omega}_{t+1}\right), \end{aligned} \tag{A.15}$$

where

$$\Psi\left(\hat{\omega}_{t+1}\right) \equiv \left(\frac{\partial H^d\left(\hat{\omega}_{t+1}\right)}{\partial\hat{\omega}_{t+1}}\frac{\omega_{t+1}}{H^d\left(\hat{\omega}_{t+1}\right)}\right)\left(\frac{d\hat{\omega}_{t+1}}{dg_L}\frac{1 + g_L}{\hat{\omega}_{t+1}}\right) > 1.$$

With (A.15) and $L_{t+1} = (1 + g_L)L_t$ one obtains (4.4). ■

A.12 Proof of Proposition 5.1

From Proposition 3.2 the steady state has $k_t = k^* > \bar{k}_c > \underline{k}_c$ so that Proposition 3.1 implies $\omega_t = \hat{\omega}^* = \omega\left(k^*\right) > \alpha$. Then, from (2.12) I have $q_t = q^* = q(\hat{\omega}^*) > 0$, and the results listed under a) - d) follow from Proposition 2.1, Proposition 2.2, Proposition 2.4, Proposition 3.1, and equations (2.13), (3.2) and (3.6). ■

A.13 Proof of Corollary 5.1

I show how a change in μ and g_L affects $\hat{\omega}^*$. Then, the corollary follows since $\partial q(\hat{\omega}^*)/\partial\omega_t > 0$ (see Proposition 2.1).

Consider the labor market equilibrium condition (3.5) and the capital market condition (3.7) in steady state. Solving both equations for k^* and substitution delivers

$$\frac{\mu\beta\left(\Gamma(1 - \gamma)\right)^{\frac{1}{\gamma}}}{(1 + \mu\beta)(1 - \nu)(1 + g_L)} = \left(\alpha\hat{\omega}^*\right)^{\frac{-1}{2}}\left(2\sqrt{\alpha\hat{\omega}^*} - \alpha\right)^{\frac{1}{\gamma}}. \tag{A.16}$$

The right-hand side of equation (A.16) defines a continuous function $RHS(\omega)$ with $RHS'(\omega) > 0$ for all $\hat{\omega}^* > \alpha$. Then, total differentiation of (A.16) delivers $d\hat{\omega}^*/d\mu > 0$ and $d\hat{\omega}^*/dg_L < 0$. ■

A.14 Proof of Corollary 5.2

The sign of $dg^*_{GDP}/d\mu$, $dg^*_{gdp}/d\mu$, and dg^*_{gdp}/dg_L follow immediately from Corollary 5.1. To show that $dg^*_{GDP}/dg_L > 0$ observe that

$$\frac{dg^*_{GDP}}{dg_L} = (1-\nu)\left(1+q^*\right)^{-\nu} \frac{dq\left(\hat{\omega}^*\right)}{dg_L}(1+g_L) + \left(1+q^*\right)^{1-\nu},$$

where $dq\left(\hat{\omega}^*\right)/dg_L = \left(\partial q\left(\hat{\omega}^*\right)/\partial\omega\right) \cdot \left(d\hat{\omega}^*/dg_L\right) < 0$. Some algebraic manipulations using

$$\frac{d\hat{\omega}^*}{dg_L} = -\frac{\hat{\omega}^*}{1+g_L}\left[\frac{1}{2} + \frac{1}{\gamma}\left(\frac{\sqrt{\hat{\omega}^*}}{2\sqrt{\hat{\omega}^*} - \sqrt{\alpha}}\right) - \nu\right]^{-1} < 0$$

obtained from (3.5) deliver the desired result. ■

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B Online Appendix: Additional Results

B.1 Empirical Evidence on Population Aging

This section provides empirical evidence on population aging for a sample of 27 OECD countries. This evidence supports the way population aging is modelled in the theoretical analysis of the main text.

As explained in Footnote 21 below, I selected those countries from the set of 36 OECD countries that experienced population aging over the time span 1960-2017. I judge this to be the case if the experienced change in the OADR is at least equal to 1. This eliminates Ireland. Moreover, due to data limitations for per-capita GDP in the Penn World Tables for the considered time span the Czech Republic, Estonia, Hungary, Latvia, Lithuania, Poland, Slovak Republic, and Slovenia are not included. Hence, the sample includes Australia, Austria, Belgium, Canada, Chile, Denmark, Finland, France, Germany, Greece, Iceland, Israel, Italy, Japan, Luxembourg, Mexico, Netherlands, New Zealand, Norway, Portugal, South Korea, Spain, Sweden, Switzerland, Turkey, UK, and the US.

Figure B.1 documents the substantial increase in the survival probability for males to age 65, a proxy for longevity, for the period 1960 - 2017. The data are from United Nations (2019). The survival probability for males to age 65 is defined as the percentage of a cohort of newborn male infants that would survive to age 65, if subject to age specific mortality rates of the specified year. The parameter μ is the counterpart to the survival probability in the theoretical analysis of the main part of this paper. This motivates the choice of this variable as a proxy for longevity.

Regressing the survival probability on a country fixed effect and years gives a slope coefficient of roughly 0.43%. Hence, over 30 years the increase in the average survival probability is 12.9%. Qualitatively similar evolutions obtain for women.

Figure B.2 shows the decline in the total fertility rate in the same sample over the same time span. The total fertility rate represents the number of children that would be born to a woman if she were to live to the end of her childbearing years and bear children in accordance with age-specific fertility rates of the specified year. Regressing the total fertility rate on a country fixed effect and years gives a slope coefficient of roughly -0.029. Hence, over 30 years the decline in the average total fertility rate is equal to -0.87.

B.2 The Evolution of the Old-Age Dependency Rate, Growth, and Factor Shares

Throughout, my stylized measure of population aging is the increase in the old-age dependency ratio (OADR). Figure B.3, Figure B.4, and Table B.1 show for 27 selected OECD countries over the time span 1960-2017 that the association between population aging and per-capita GDP growth is positive and significant whereas the association with the

Figure B.1: The Increase in the Survival Probability for Males to Age 65 in 27 Selected OECD Countries from 1960-2017.

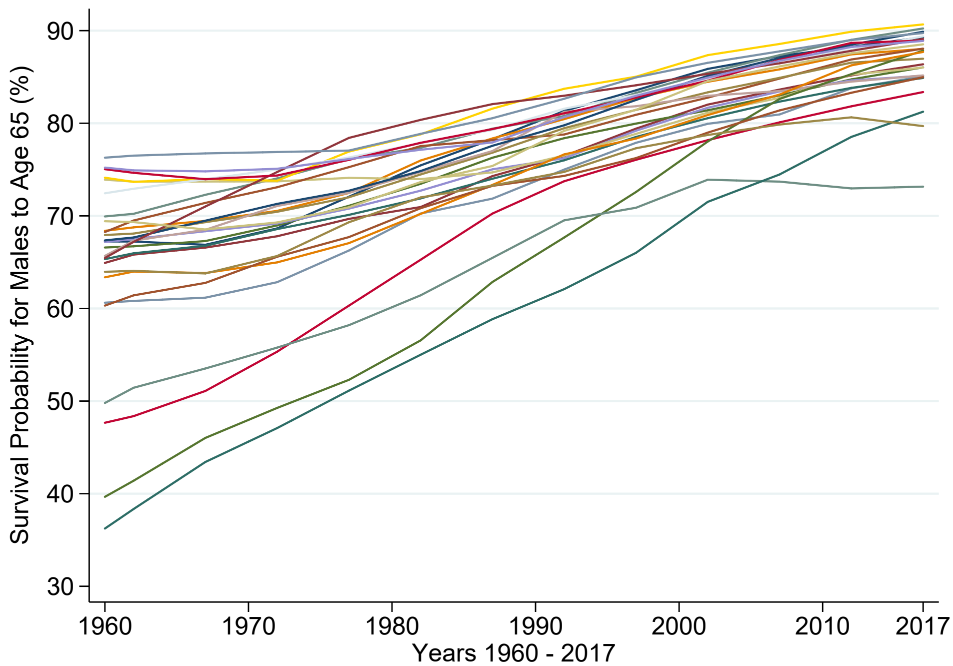
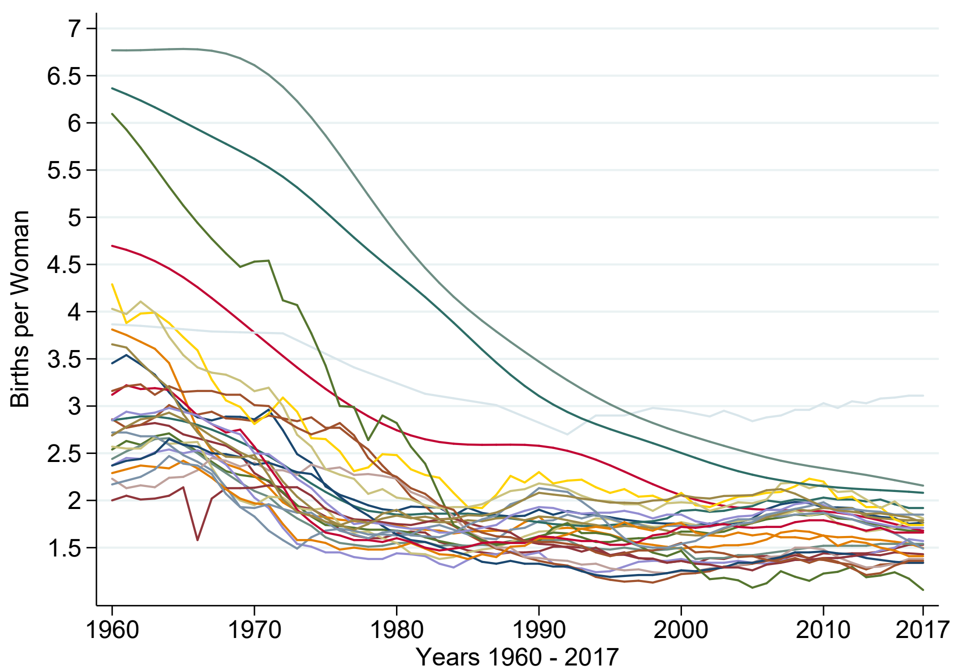
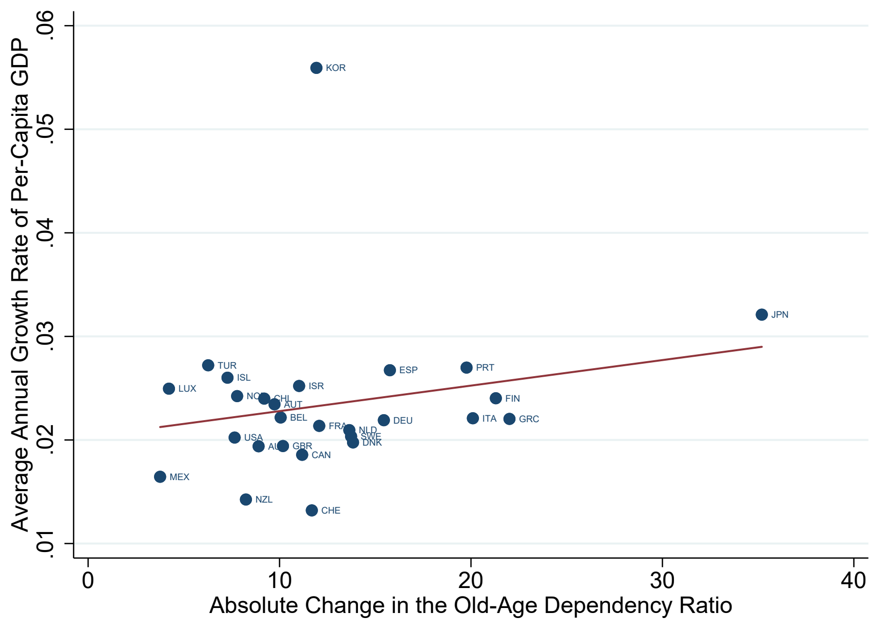


Figure B.2: The Decline in the Total Fertility Rate in in 27 Selected OECD Countries from 1960-2017.



change in the labor share is negative and insignificant.[21] Table B.1 also suggests that the order of magnitude of the association between aging and growth is not negligible. Of two otherwise identical countries in 1960 the one for which the increase in the OADR is higher by 10 units is predicted to have a 15% higher level of per-capita GDP in 2017. This raises the question about the role of automation for these trends.

Figure B.3: Population Aging and the Average Annual Growth Rate of Per-Capita GDP 1960-2017 for 27 Selected OECD Countries.



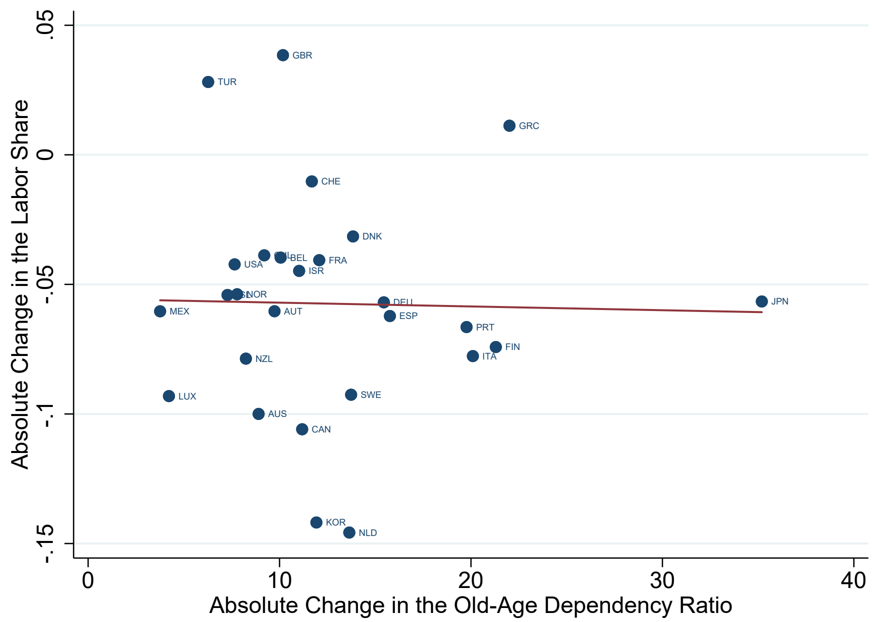
Note: Figure B.3 shows the positive and significant association between population aging and per-capita GDP growth 1960-2017. The data for GDP and population are extracted from the Penn World Table 9.1 (Feenstra, Inklaar, and Timmer (2015)). Data for the OADR are taken from the World Bank (United Nations (2019)). The OADR states the dependent population aged 65 and older per 100 members of the working population between 15 and 64 years of age.

B.3 A Simple Calibration Exercise

This section shows that a simple calibration of the steady state of Proposition 3.2 delivers reasonable results. A period corresponds to 30 years. The calibration delivers an annual steady-state growth rate of per-capita output, per-capita consumption and savings of 2%, an annual rate of decline in the individual supply of hours worked of 0.66%, and a labor

[21] From the set of 36 OECD countries I select countries that experienced population aging over the time span 1960-2017. I judge this to be the case if the change in the OADR over the considered time interval is at least equal to 1. This eliminates Ireland. Moreover, due to data limitations for the entire time span 1960-2017 in the Penn World Table the Czech Republic, Estonia, Hungary, Latvia, Lithuania, Poland, the Slovak Republic, and Slovenia are not included.

Figure B.4: Population Aging and the Change in the Labor Share 1960-2017 for 27 Selected OECD Countries.



Note: Figure B.4 shows the negative, yet, insignificant association between population aging and the change in the labor share 1960-2017. The data for the labor share are extracted from the Penn World Table 9.1 (Feenstra, Inklaar, and Timmer (2015)). Data for the OADR are the same as those used in Figure B.3.

share of slightly less than 2/3. Throughout, I round resulting numbers to the second digit after the decimal point.

On the *production side* I set $\Gamma = 6.15$, $\gamma = .25$, and $\alpha = 1$. The following parameter values are chosen for the *household sector*:

$$\mu = 70\%, \quad \beta = \frac{10}{21}, \quad \nu = \frac{1}{4}, \quad \phi = \frac{1}{2}\left(\frac{3}{2}\right)^{\frac{1}{3}}, \quad \text{and} \quad g_L = 35\%.$$

I proxy μ with the probability at birth for males of reaching the age of 65 as shown in Figure B.1. In line with the literature, the chosen value for β corresponds to an annual discount factor of roughly 0.976 (see, e. g., Prescott (1986), Blanchard and Fischer (1989), p. 147, or Barro and Sala-í-Martin (2004), p. 197). The wage elasticity of hours worked, ν, is in line with the value suggested by Boppart and Krusell (2020). The preference parameter ϕ is chosen such that $w_c = 1$. Then, Assumption 1 holds for $w_t > 1$ as $\nu = 1/4 < \bar{\nu}(\mu\beta) = 0.35$. Finally, the fertility rate, g_L, implies that cohorts grow at an annual rate of 1%.

Table B.1: Estimates of the Impact of Population Aging on the Average Annual Growth Rate of Per-Capita GDP (Panel A) and the Change in the Labor Share (Panel B) 1960-2017 for 27 Selected OECD Countries.

Panel A. Estimates of the Impact of Aging on Per-Capita GDP Growth	
Change of the OADR	0.00025
	(0.0001)
Observations	27
R^2	0.0479
Panel B. Estimates of the Impact of Aging on the Labor Share	
Change of the OADR	−0.00033
	(0.0009)
Observations	27
R^2	0.025

Note: Robust standard errors in parentheses. Panel A, shows the positive and significant association between population aging and per-capita GDP growth 1960-2017. Panel B, shows the negative, yet, insignificant association between population aging and the change in the labor share 1960-2017.

Proposition B.1 *(Steady State of the Calibrated Economy)*

Let $A_0 > 1$ and suppose that the calibrated economy embarks on a steady state in $t = 1$. Then, the steady state satisfies $w_t > \alpha A_{t-1} > w_c$ for all $t = 1, 2, ..., \infty$. Moreover, it holds that

$$k^* = 0.45, \quad \omega^* = 4.88,$$

and

$$q^* = 1.21, \quad g^*_{h^s} = -0.17, \quad LS^* = 0.66, \quad R^* = 3.78.$$

Proof of Proposition B.1

Consider the labor and the capital market.[22] For the chosen parameter constellation $\Lambda = 452.632$ and $\underline{k}_c = 0.0022093$. Moreover, the labor-market equilibrium condition (3.5) reads

$$k_t = 0.0022093\,(2\sqrt{\omega_t} - 1)^4\,\omega_t^{\frac{1}{4}}. \tag{B.1}$$

At the same time, $\Omega = 0.246914 = \overline{k}_c$. Hence, it holds that $\overline{k}_c = 0.246914 > \underline{k}_c = 0.0022093$, and the capital market equilibrium condition (3.7) becomes

$$k_{t+1} = 0.246914 \cdot \omega_t^{\frac{3}{8}}. \tag{B.2}$$

[22]All computations were executed in *Mathematica*. The relevant notebooks are available upon request.

Equations (B.1) and (B.2) determine the equilibrium difference equation (3.8) as

$$k_t = 0.00561338 \cdot k_{t+1}^{2/3} \left(12.9113 \cdot k_{t+1}^{4/3} - 1\right)^4 . \tag{B.3}$$

The evaluation of (B.1) and (B.2) at $k_t = k_{t+1} = k^*$ and $\omega_t = \omega^*$ delivers k^* and ω^* as stated in the proposition. Since $\omega^* > 1$ and $\alpha = 1$, the steady state satisfies $w_t > A_{t-1}$ for all $t = 1, 2, ..., \infty$. Since $w_c = 1$ it also satisfies $\alpha A_{t-1} > w_c$ if $A_{t-1} > 1$.

Using ω^* in Proposition 2.1 delivers the indicated value of q^*, using ω^* in (**??**) gives the stated labor share, LS^*. With ω^* in Proposition 2.1 one also finds

$$i^* = 1.20823 \quad \text{and} \quad c^* = 3.41645.$$

With the latter in Proposition 2.2 one obtains

$$\frac{N_t}{K_t} = \left(\frac{\Gamma(1-\gamma)}{c^*}\right)^{\frac{1}{\gamma}} = 3.32234.$$

For $N_t/K_t = 3.32234$ the first-order condition for K_t in (2.13) gives R^*.

Finally, from Proposition 2.1 the growth factor of the supply of hours worked is $1/(1+q^*)^\nu = 0.820331$. ■

To interpret these numbers observe that $q^* = 1.21$ implies an annual growth rate of 2.68% for the labor productivity per task and for the real wage. Moreover, from Proposition 5.1, the growth factor of per-capita output, per-capita consumption and savings satisfies $(1+q^*)^{1-\nu} = 2.21^{3/4}$ which implies an annual growth rate of 2%. The growth rate of the individual supply of hours worked, $g^*_{h^s}$, corresponds to an annual growth rate of -0.66%.[23] The order of magnitude of the labor share is also in line with the empirical evidence.

Finally, as fixed capital fully depreciates the real rate of return on capital is $R^* - 1$ which corresponds to an annual rate of 4.54%.[24]

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[23]For the US the PWT 9.0 estimates average annual hours per person engaged in 1960 to equal 1863. In 2010 the corresponding number is 1695 (Feenstra, Inklaar, and Timmer (2015)). This corresponds to an average annual growth rate of -0.19%. The estimates of the annual hours worked per worker of Huberman and Minns (2007) for the US are much higher than the numbers in the PWT 9.0. According to Huberman and Minns (2007) annual hours worked per worker in 1960 was 2033 whereas this number plunges to 1878 for the year 2000. However, the implied average annual growth rate of roughly -0.18% is in line with the one found for the PWT data. The calibration chosen here implies an average annual growth rate of hours worked of -0.66% which is closer to the estimate that Boppart and Krusell (2020) derive for the sample of countries included in Figure B.1 over the time span 1870-2000. This confirms the view that the US evolution of hours worked per worker is an outlier.

[24]Piketty (2014) asserts that the real rate of return on capital often exceeds the economy's growth rate. In my notation this means that $R^* - 1 > (1+q^*)^{1-\nu}(1+g_L)$. With the numbers of Proposition B.1 this inequality holds since $2.78 > 2.45$.