Leisure-Enhancing Technological Change*

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Abstract

Modern economies are awash with leisure-enhancing technologies: products supplied in exchange for time and attention, rather than money. This paper studies where such technologies come from and how they interact with the broader macroeconomy. Leisure innovation arises endogenously, propelled by the desire to capture consumers' time and attention – inputs used to produce intangible assets such as brand equity capital. The non-rival nature of leisure products implies that the sector that is small in the aggregate can have significant effects at a macro level. The indirect monetization of leisure technologies means that the equilibrium level of leisure-focused R&D is inefficient.

Keywords: time allocation, leisure, platforms, productivity, non-rivalry, zero prices.

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1 Introduction

In models of economic growth, technological change is a catch-all generalization of a large and diverse set of innovations undertaken in the real world. In this paper I distinguish between "traditional" product- or process-innovations and inventions that are *leisure-enhancing*.

The defining features of leisure technologies are that they (i) complement leisure time, (ii) are non-rival, and (iii) tend to be indirectly monetized. Indeed, leisure-enhancing products are often available at the marginal cost of zero, and are profitable precisely because they capture consumers' time and attention – resources that are greatly valuable in the modern economy. The main insight of this paper is that traditional and the leisure-enhancing technologies interact in ways that shed new light on important macroeconomic phenomena, such as dynamics of hours worked and productivity.[1]

Consider social media as a telling example. Survey estimates suggest that in 2020 an estimated 4 billion active users have spent on average over 2 hours a day using social media.[2] This success in terms of capturing consumers' time appears to have been achieved, in part, by innovation activity in the social media sector. Consumers can tap into social media services without reaching for their wallets: it is their time, attention and data that buys them access.[3]

These salient features carry beyond the social media platforms operating in recent years. The "leisure-R&D sector" is an important cluster of innovation and discovery. For example, a proxy for its share in overall R&D spending across the industrialized world has more than doubled between 2005 and 2014, according to the data produced by the OECD.[4] Monetizing time and attention is not a new phenomenon: the left panel of Figure 1 shows that indirectly financed zero-price products go back decades. Moreover, the literature suggests that leisure technologies have been instrumental in shifting time allocation patterns: for example, Aguiar and Hurst (2007a) and Gentzkow (2006) find evidence that the introduction of the television in the 1950s and 1960s had a large

[1] Thus, these technologies offer a new perspective on the Solow Paradox. In 1987 Bob Solow famously quipped that "computer age is visible everywhere except for the productivity statistics". Computer age eventually made an appearance in the mid-90s, driving much of the pick-up in growth in capital intensity and total factor productivity in the United States (see Jorgenson (2005) for a summary). This revival was ultimately short-lived, and TFP growth since the early 2000s has again been puzzlingly sluggish. The perception of rapid technological change appears to be, once again, at odds with the official statistics.

[2] The figures are from Globalwebindex, a consultancy which runs a large survey of online behaviors.

[3] Industry estimates suggest that over 90% of social media firms' revenues comes from advertising (OfCOM, 2019). In this paper attention- and data gathering are assumed to be perfectly correlated with capturing consumers' time. Nonetheless, the distinction may play an important role in the context of the modern technologies. Complementary work of Farboodi and Veldkamp (2019) considers the long-run consequences of data gathering.

[4] See Figure A.2 in Appendix A.

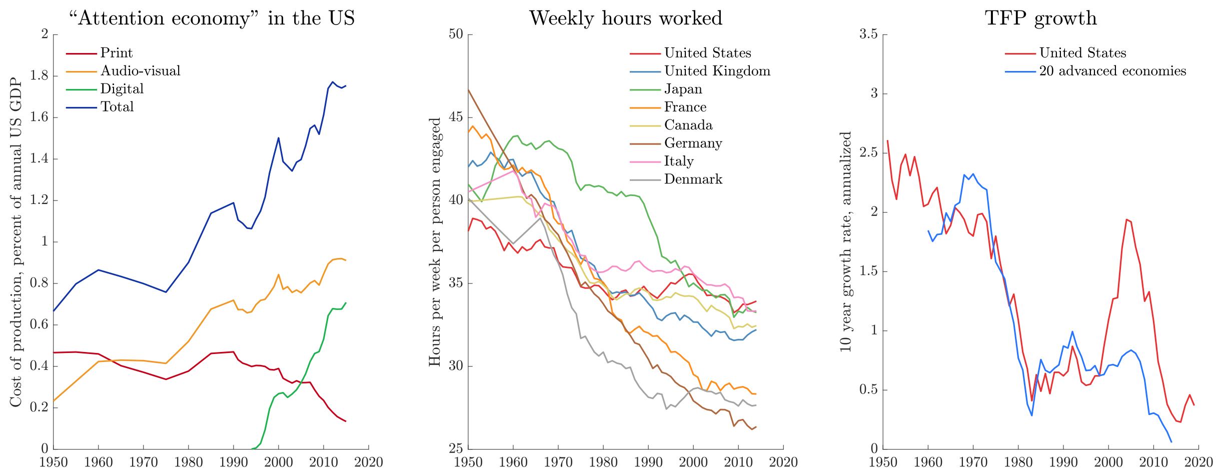

Figure 1

Motivating Trends: Free Products in the United States, and Cross-Country Trends in Hours Worked and Total Factor Productivity

Notes: Estimates of the cost of production of free consumer services are from the Bureau of Economic Analysis (Nakamura *et al.* (2017)). The figure shows the ratio of free consumer content, measured by the costs of production, to GDP. Thus, for example, it does not attempt to capture utility benefit of Facebook, but only the cost of providing it. Hours worked are from Penn World Tables 9.0 (Feenstra *et al.* (2015)). The US TFP growth rate is the utilization-adjusted series following Basu *et al.* (2006). The TFP growth rate for advanced economies is constructed by the IMF and is PPP-weighted (Adler *et al.* (2017)). Both series show 10-year growth rates.

impact on time allocation in the United States, and Falck *et al.* (2014) document the significant impact on leisure time of the roll-out of the internet in Germany in the 2000s. Both episodes constituted an expansion of free-of-charge, ad-financed services available to consumers.

The technological developments in leisure have occurred against the backdrop of a trend decline in hours worked (Figure 1, middle panel) and slowing growth of labor- and total factor productivity (the right panel). How, if at all, are these trends linked? To tackle this question I proceed in three steps.

First, I show how to tractably incorporate zero-price leisure technologies in individuals' time allocation decisions, and tease out how the associated choices alter the long-run growth dynamics. The novel feature in this framework is that individuals derive leisure utility from various activities such as watching TV or browsing the web. These activities require both time and market inputs such as TV channels, web pages, mobile apps, etc., which I assume are available to all households for free, i.e. at no monetary cost. I show that if leisure products are complementary with leisure time, in the sense that better leisure products raise the marginal utility of leisure time, then improvements in leisure technology drive increases in the amount of time households dedicate to leisure. If, as in the canonical endogenous growth theory, the development of traditional technologies relies on human input, there is a negative spillover to traditional productivity growth.

These insights rely on the complementarity in utility between leisure time and leisure technology. Further restrictions are needed for this process to be consistent with *balanced* growth. I characterize the class of utility functions that is necessary and sufficient to deliver balanced growth in a market equilibrium with (exogenous or endogenous) zero-price leisure technologies, and in the optimal allocation where the planner optimally trades-off the use of resources in the traditional and in the leisure sectors. I explain how the presence of zero price products in equilibrium means that the conditions for balanced growth are more stringent in the optimal allocation, compared to the equilibrium.

Where do these zero-price leisure-enhancing technologies come from, and does the market do a good job at providing them? To answer these questions, in the second step of the analysis I build a (static) model of an *attention economy* – an economic ecosystem that supports the existence of leisure-enhancing innovations. There are three elements to the framework. First, firms in this economy compete not only on prices but also by advertising their products. They do so by investing in brand equity – a form of intangible capital that is all about the recognition in the minds of prospective customers. Second, such intangible capital must itself be produced, and consumers' time (and, more loosely but relatedly, attention and data) are the crucial inputs in this process. Third, leisure technologies can "capture" these inputs, allowing leisure-innovators ("platforms") to engage in the production of brand equity capital.

A salient feature of leisure products is their non-rivalry: once a leisure technology is invented, it can be enjoyed by any number of consumers. Indeed, because these ideas are used directly by consumers, they exhibit a strong form of non-rivalry, in the sense that the marginal cost of supplying an additional user with an existing leisure variety (a social media site, say) is zero.[5] Despite the presence of increasing returns – a costly invention process coupled with a zero marginal cost of provision – an equilibrium with pricing at marginal cost can exist. I illustrate this with a framework that assumes there exists a competitive fringe that can imitate any existing leisure product at (zero) marginal cost, thus precluding equilibrium prices above the marginal cost of zero. Such lack of property rights would normally destroy the incentives to innovate in the first place. Yet the ability to indirectly monetize leisure technologies – generating revenue from brand equity sales – allows platforms to recoup their costs.[6]

While the indirect monetization allows some provision of leisure technologies in equilibrium, it does not guarantee that this provision is efficient. In equilibrium private

[5] This is unlike most "traditional" ideas which are embodied in products that require real resources to produce; e.g. a drug recipe is non-rival and yet the actual ingredients to produce the drug might be expensive.

[6] This ability to recoup costs is there as long as the fringe does not have the ability to generate and sell brand equity. This highlights that while marginal cost pricing of leisure innovation is possible in equilibrium only if imperfect competition is also there to allow firms to recoup upfront innovations costs.

returns to leisure innovation are tied to the demand for brand equity and ignore the direct utility benefits these technologies bring to consumers.

The third and final step of the analysis combines the dynamic model of growth with the model of the attention economy. The first insight from the full model is that leisure-enhancing technologies emerge endogenously on the growth path once the economy becomes sufficiently large.

I characterize analytically the steady state growth rates of traditional and leisure technologies, hours worked and per-capita consumption on both segments of the sBGP, pre- and post- the first entry of the platforms. Leisure technology growth lowers the long-run growth rate of per capita consumption directly, by driving the declining trend in hours worked, and indirectly, through its impact on the growth of traditional technology. Moreover, the share of resources devoted to leisure R&D rises at the expense of the share devoted to traditional R&D. This acts to depress traditional growth along the transition, and has persistent level effects. I study the efficiency properties of the dynamic equilibrium and show how leisure technologies interact with the well-understood inefficiencies present in economies with endogenous growth.

To give a sense of the magnitudes of the macroeconomic effects, I present an illustrative quantification. The key takeaway from this exercise is that despite representing a small fraction of the aggregate economy the leisure R&D sector can have significant effects on the growth pattern: the emergence of the attention economy in post-war decades can plausibly account for up to a half of the slowdown in traditional TFP growth observed over the past 70 years.

Finally, the theory can help better understand the measurement challenges associated with zero-price products. These products are not included in GDP as currently measured. Two questions arise: first, does this mean that GDP is *significantly* mismeasured? And second, is GDP becoming a less reliable guide to *welfare*? I answer the first question in the negative: consistency with how other components of GDP are measured requires measuring the value of leisure technologies at the cost of production, which is relatively small in the data. But, thanks to the non-rivalry, the value of these technologies from the users' perspective can be much greater than the cost of their production, so that GDP does indeed miss a welfare effect of leisure technology improvements: along the sBGP, the leisure-enhancing technologies introduce a wedge between GDP and welfare that becomes proportionately larger over time.[7]

[7] These findings suggest that leisure time (enhanced by leisure technology) ought to be included in measures of economic wellbeing, in the spirit of Nordhaus and Tobin (1972) and Stiglitz *et al.* (2009).

Related literature. In proposing a directed-technology explanation for the trend in hours worked, this paper brings together the literatures on endogenous innovation[8] with that on the long-run shifts in time allocation.[9] Since the seminal paper of King *et al.* (1988) which derive the 'balanced growth' preference class, most growth models have featured constant hours worked along the balanced growth path.[10] Yet the historical data which show a steady long-run decline of around -0.4% per annum (Jones, 2015).[11] In contributions closely related to this paper, Ngai and Pissarides (2008) and Boppart and Krusell (2020) provide two alternative accounts for this trend: the former paper highlights the role of differential sectoral growth rates and non-separability of preferences while the latter characterizes the preference class that delivers an income effect larger than the substitution effect along the BGP. Both of these papers and other related contributions assume growth is exogenous. Instead, this paper assumes separable balanced growth preferences and instead focuses on the endogenous rise of the attention economy.

The present paper extends the line of research recently summarized in Aguiar and Hurst (2016) which develops a unified theory of consumption and time allocation. The contribution is to develop a tractable model for analysis of zero price services. The focus on leisure technologies brings the paper close to Aguiar *et al.* (2017) who investigate how video games have altered the labor supply of young men in the United States. Relative to that paper I cast the net more broadly.[12]

The paper also contributes to the literature on the productivity slowdown and the mismeasurement hypothesis.[13] It shows that while mismeasurement of GDP (a production-based metric) is second order, a growing disconnect between GDP and measures of economic wellbeing is likely.

Finally, this paper builds on the literature on two-sided markets, intangible capital

[8] Romer (1990), Aghion and Howitt (1992a), Jones (1995), Kortum (1997), Segerstrom (1998).

[9] Prominent contributions include Aguiar and Hurst (2007a), Ramey and Francis (2009), Aguiar *et al.* (2017), Vandenbroucke (2009), Aguiar *et al.* (2012) and Scanlon (2018).

[10] Some papers have introduced trends directly into preferences. Scaling leisure utility by a term that increases at a rate proportional to technology and wages and doing so in a way such that utility is homogenous in technology yields a balanced growth path with utility functions that fall outside of the King *et al.* (1988) class. See Mertens and Ravn (2011) and Guvenen and Rendall (2015) for example of this approach.

[11] Leisure inequality has increased as poorer households increased their leisure time by more than the rich (Aguiar and Hurst (2008), Boppart and Ngai (2017a)). The free leisure technologies could be important in helping to explain this divergence. Investigating this hypothesis is left for future work.

[12] The present paper speaks to historical events such as the roll-out of the TV in the 1950s as well as the more recent digital trends and considers the whole swathe of free technologies which are used by a vast majority of the population, whereas Aguiar *et al.* (2017) focus on computer games which are used primarily by young men. This paper also goes beyond the labor supply effects and explores the implications for total factor productivity, measurement and welfare.

[13] Useful references include Brynjolfsson and Oh (2012), Byrne *et al.* (2016a), Bean (2016), Bridgman (2018), Syverson (2017), Coyle (2017), Aghion *et al.* (2017), Nakamura *et al.* (2017), Hulten and Nakamura (2017), Brynjolfsson *et al.* (2018) and Jorgenson (2018).

and advertising in industrial organization and in macroeconomics.[14] Relative to these literatures its contribution is to study the consequences of how intangible assets are produced.

Roadmap. Section 2 begins by introducing the zero-price leisure technologies in the household problem and explores the implications for growth of traditional productivity. Section 3 derives the restrictions on preferences that are consistent with balanced growth when there is leisure technology growth. Section 4 contains the model of the attention economy. Section 5 presents the full dynamic model. Section 6 presents the parametrization exercise. Section 7 discusses measurement. Section 8 concludes.

2 Leisure technologies and time allocation decisions

This section develops a tractable way to incorporate zero price leisure technologies into consumer's time allocation problem, and provides a first look at the spillovers from leisure technology to traditional technology.[15]

2.1 Setup

The economy is populated by measure N of infinitely-lived households. Population size increases at rate n. Households discount the future at rate ρ and derive utility from per-capita consumption c and from leisure l with an instantaneous utility function of the form $u(c, l) = \log c + l$. For now, the advantage of this formulation is that it allows for an intuitive microfoundation for how leisure technologies feature in the households' problem. The next section discusses which properties of this utility function are crucial for the results of this paper, and characterizes a general preference class that shares these properties.

The main departure from the textbook model is that, to generate leisure utility, each individual engages in *leisure activities*, such as watching TV or browsing the web. Each activity requires *time* and *access to a complementary product*, e.g. a television channel

[14] Classic references on the economics of platforms are Rochet and Tirole (2003) and Anderson and Renault (2006) who study the equilibrium pricing decisions in two-sided markets. A useful review of the literature is provided by the Handbook Chapter by Bagwell (2007). Several papers analyzed theoretically the way in which ads enter the consumer problem, and what the positive and normative implications are (Dorfman and Steiner (1954), Dixit and Norman (1978), Becker and Murphy (1993), Benhabib and Bisin (2002)) as well as the businesses decisions to invest in and accumulate intangible capital (Hall (2008), Corrado and Hulten (2010), Corrado *et al.* (2012), Gourio and Rudanko (2014),Cavenaile and Roldan-Blanco (2020)).

[15] The production side and the intertemporal problem of the representative consumer follow a textbook treatment, so the focus here is on household's intratemporal decisions.

or a social media site. There is measure M of products (and hence activities), indexed by ι and available to each household at no monetary cost.[16]

Each individual solves the following problem:

$$\max_{c(t),h(t)\in[0,1],\{\ell_\iota(t)\}_0^{M(t)}} \int_0^\infty e^{-\rho t}\left(\log c(t) + l(t)\right) dt \quad \text{subject to} \tag{1}$$

$$\dot{k}(t) = w(t)h(t) + r(t)k(t) + \hat{\pi}(t) - c(t) \tag{2}$$

$$l(t) = \left(\int_0^{M(t)} \left[\ell_\iota(t)\right]^{\frac{1}{1+\zeta}} d\iota \right)^{1+\zeta} \tag{3}$$

$$\ell(t) = \int_0^{M(t)} \ell_\iota(t) d\iota = 1 - h(t) \tag{4}$$

$$k(t) \geq 0, \;\; k_0 \text{ given.} \tag{5}$$

Just as in the neoclassical growth model, households save in capital k with net return r and earn hourly wage w (both r and w are taken as given); they own the firms and earn unearned income $\hat{\pi}$ each period. Households choose labor supply h and decide how to allocate their leisure time ℓ across the available activities (time endowment is normalized to 1).[17] The important parameter $\zeta \geq 0$ indexes the strength of the love of variety effect (the elasticity of substitution across activities is given by $1 + \frac{1}{\zeta}$). Finally, the range of available activities may change over time – in particular, for now assume that $M(t)$ increases exponentially at an exogenously given rate $\gamma_M \geq 0$.[18]

The index of traditional technology $A(t)$ expands through profit-driven horizontal innovation, as in Romer (1990) and Jones (1995). Specifically, ideas are developed by researchers, with a success rate that depends on the existing stock of knowledge:

$$\dot{A}(t) = L_A(t)A(t)^\phi, \tag{6}$$

where $L_A(t) := s_A(t)h(t)N(t)$ is labor employed in generating ideas and $s_A(t)$ is the share of labor employed in the R&D sector. Parameter ϕ guides the strength of knowledge

[16] Appendix G explores a more general formulation with consumption goods as inputs into leisure activities.

[17] Activities that do not involve free leisure technologies, such as walking in a park or hiking, are outside of the benchmark model, but are straightforward to incorporate (see Appendix H). Similarly, the baseline model abstracts from home production for simplicity, and focuses on the leisure vs. market hours margin. Incorporating home production is left for future research.

[18] Throughout the paper I assume that the discount rate ρ is sufficiently high to ensure that household utility is finite. In the context of the model of this section, we require $\rho > \zeta\gamma_M$. I also rule out the cases with zero or negative per-capita steady state consumption growth. This requires $\frac{n}{2-\phi} > \zeta\gamma_M$ here; in the full model with endogenous γ_M it is sufficient to assume that $\zeta \in [0, 1)$.

spillovers; I assume that $\phi < 1$.[19]

While it is not necessary to specify the full structure here, in the background one can think of the final consumption good (numeraire) being produced by competitive firms with a constant-returns production function, combining intermediate inputs x_i of measure $A(t)$ with labor $L_Y(t) := (1 - s_A(t))h(t)N(t)$:

$$Y(t) = \int_0^{A(t)} x_i(t)^\alpha L_Y(t)^{1-\alpha} di. \tag{7}$$

Monopolistically competitive intermediate producers use capital with a one-to-one linear technology: $x_i = k_i \forall i$. In this setting, wages grow in line with $A(t)$ on the BGP.

I offer a more complete description of the environment in subsequent sections and instead focus here on characterizing the solution to the representative individual's time allocation problem.[20]

2.2 Equilibrium time allocation

The time allocation problem consists of two stages: (1) allocate time across leisure activities; (2) allocate time between work and leisure. In the first stage, given the symmetry of the problem, it is optimal to spend an equal share of leisure time on each activity:

$$\ell_\iota(t) = \ell(t)/M(t). \tag{8}$$

Substituting (8) into (3) gives the derived instantaneous utility function

$$u(c, h, M) = \log c(t) + M(t)^\zeta (1 - h(t)). \tag{9}$$

The second term in (9) is leisure utility at every instant. The key point here is that leisure utility depends not only on leisure time, but also on the index of leisure technology. The activity-based framework severs the tight link between leisure time and leisure utility embedded in a standard model.

The interior labor-leisure choice satisfies the usual optimality condition that the utility benefit of working a little more must equal the utility cost: $u_c \cdot w = -u_h$. Applying this

[19] This places my benchmark framework within the semi-endogenous class of growth models (Jones, 1995). The evidence does indeed suggest that ideas "are getting harder to find", supporting the assumption of $\phi < 1$ (Bloom *et al.*, 2020). But the lessons here are more general and extend beyond this particular underlying growth paradigm.

[20] The market equilibrium is defined in a standard way. I defer the definition of the equilibrium to the full model with endogenous M below.

to (9) and noting that hours worked cannot exceed the time endowment of 1 gives:

$$h(t) = \min\left\{1, \Phi(t)M(t)^{-\zeta}\right\}, \tag{10}$$

where $\Phi(t) := \frac{w(t)h(t)}{c(t)}$ is the ratio of labor income to consumption. Equation (10) says that on a balanced growth path (with $M(t)$ sufficiently high), households spend some of their time on leisure activities. Moreover, in that case and as long as $\zeta > 0$, improvements in leisure technology – a higher M – lead to a decline in hours worked. This result rests on the complementarity between leisure time and leisure technology in utility: when ζ is strictly positive, the cross-derivative $u_{\ell M} = \zeta M^{\zeta-1}$ is positive, meaning that a higher M raises the marginal utility of leisure time. Furthermore, the budget constraint in (2) implies that the ratio $\Phi(t)$ is constant on the BGP. Thus, taking logs and differentiating (10) with respect to time, we obtain that the growth rate of hours worked in steady state denoted with γ_h is

$$\gamma_h = -\zeta\gamma_M. \tag{11}$$

Equation (11) shows that *exponential* growth in M results in an *exponential* decline in hours. This exponential, or "balanced", nature of this process comes about because of a balanced growth property of the utility function in (9). Section 3 discusses these properties in detail and characterizes a general class that possesses such balanced growth property.

2.3 Growth effects of leisure technologies

Leisure technology has a broader macroeconomic impact through the effects on the time allocation patterns. Since the share of labor employed in the R&D sector s_A is constant on a balanced growth path, differentiating equation (6) with respect to time we obtain the following expression for the long-run growth of traditional technology:

$$\gamma_A = \frac{n + \gamma_h}{1 - \phi}. \tag{12}$$

Combining (11) and (12) gives the following result (all proofs are in the Appendix):

Proposition 1. *Assume that $n > \zeta\gamma_M$. Then the economy converges to a balanced growth path with hours declining at a constant rate given by (11) and $A(t)$ increasing at a constant rate γ_A given by*

$$\gamma_A = \frac{n - \zeta\gamma_M}{1 - \phi}. \tag{13}$$

The BGP growth rate of $A(t)$ is thus decreasing in γ_M. The growth rate of per-capita *consumption, denoted with , γ_c is*

$$\gamma_c = \frac{n - (2-\phi)\zeta\gamma_M}{1-\phi}. \tag{14}$$

The result is that leisure technology growth weighs down on the growth rate of output and consumption directly through its impact on the labor supply (equation (11)), and also indirectly through TFP growth (equation (13)). The long-run growth rate of A is pinned down by the growth rate of the pool of resources devoted to generating ideas. Leisure technologies reduce the growth of that pool via their impact on hours worked.

The formula in (13) is specific to the semi-endogenous growth framework underlying the analysis, but the mechanism is present in a broader class of models with expanding varieties where innovation requires real resources such as those building on Romer (1990); Grossman and Helpman (1991). A complementary interpretation is highlighted by Schumpeterian models of growth where the diminished market size for traditional innovations – due to declining hours worked and thus lower traditional consumption growth – lowers the incentives to innovate (Aghion and Howitt (1992b); Acemoglu and Linn (2004); see also Appendix I).

The takeaway from this simple model is that, through its effects on time allocation, leisure technology that complements leisure time affects the growth process and thus can have important macroeconomic effects. The balanced growth path can feature hours worked that decline at a constant rate, and this can result in a lower steady state growth rate of traditional TFP, relative to the case with no leisure technology improvements. These results were established using a specific functional form of preferences, and the analysis ignored the all-important question of where these zero-price technologies come from. The remainder of the paper addresses these questions: the next section generalizes preferences; Section 4 develops a model of endogenous leisure technology, and Section 5 studies the interactions between traditional- and leisure technology growth.

3 Leisure technology and balanced growth

The utility function in the previous section (equation (9)) is special: it is additively separable, consumption utility is log, and leisure enters linearly. Which of the underlying properties of this utility function are important for the results? And does the analysis extend to more general preferences?

The key property is that leisure time and leisure technology are complements: the cross-derivative u_{hM} is negative (equivalently, $u_{\ell M} > 0$), and consequently improvements in leisure technology lead to a rise in leisure hours and a decline in hours worked. Second, this utility function is consistent with balanced growth in equilibrium when both A and M technologies improve over time. Finally, higher M raises utility: $u_M > 0$. This is a natural property given the widespread use of these technologies and is consistent with the evidence in the literature that these technologies are valued by consumers.[21] I now proceed to derive a general class of utility functions $u(c, h, M)$ which satisfies these properties, focusing first on the balanced growth property.

3.1 Defining balanced growth preferences

Consider an economy with constant steady state growth of traditional- and leisure- technologies (these could be exogenous, or a result of endogenous efforts within the economy). We seek preferences that, given this dual expansion of the technology frontier, would deliver a balanced growth path: a long-run equilibrium along which (traditional) consumption and output grow at a constant rate. Appendix D formally defines balanced growth preferences, using the first order conditions of the relevant utility maximization problem (of the representative household in an equilibrium setting, or of a planner in a social optimum problem). These conditions relate the various marginal rates of substitution (MRS) to objects that, in the long-run, grow at constant rates determined by the growth rates of A and M. First, the intratemporal condition sets the MRS between consumption and leisure equal to the hourly wage, which grows in line with the traditional technology; second, the intertemporal condition involves the MRS between consumption across different dates, which is constant and equal to the return to wealth. If leisure technology growth is endogenous, the planner's problem features an additional third condition that pins down the optimal allocation of resources across sectors, which involves the MRS between traditional consumption goods and leisure goods. This condition is not part of the equilibrium since, in a decentralized economy with zero prices of leisure products, consumers do not face a trade-off between traditional consumption and consumption of leisure varieties.[22] The main idea behind the results in this section is that, through these conditions, BGP requirements impose restrictions on the MRSs, and hence on the utility function u. Since an additional condition involving preferences must hold in the optimal allocation, the requirements for a BGP are stricter in the social optimum than they are in the decentralized equilibrium with indirect monetization and zero prices of leisure goods.

[21] See e.g. Brynjolfsson *et al.* (2019) and references within. This property is not strictly required for the positive implications of the theory, however: for example, leisure technologies could be modeled as increasing the disutility of work, and, as long as $u_{\ell M} > 0$ (or equivalently $u_{hM} < 0$) households would choose to work less as leisure technologies "improve". Of course, if given the choice, consumers (or a social planner) would choose not to use any such technologies – in other words, such a theory would require different microfoundations and its normative implications would be rather mechanical.

[22] Section 4 shows that the equilibrium condition that pins down the sectoral allocation of resources is independent of preferences, and thus does not restrict them.

3.2 BGP-consistent preference class

The following results characterize the class of utility functions consistent with balanced growth.

Proposition 2. *Save for multiplicative and additive constants, a utility function $u(c,h,M)$ is consistent with balanced growth in a decentralized equilibrium with zero-price leisure products and with exponential growth in wages and leisure technology if and only if*

$$u(c,h,M) = \begin{cases} \frac{\left(c^{1-\epsilon}M^{\epsilon}v\left(c^{\varrho}h^{1-\varrho}M^{\zeta}\right)\right)^{1-\sigma}}{1-\sigma} + f(M) & \sigma \neq 1 \\ \log\left(c^{1-\epsilon}M^{\epsilon}\right) + \log v(c^{\varrho}h^{1-\varrho}M^{\zeta}) + f(M) & \sigma = 1 \end{cases} \tag{15}$$

for any arbitrary twice continuously differentiable functions v and f.[23]

u is also consistent with a BGP in the optimal allocation where leisure technology growth is endogenous and the planner optimally chooses the share of resources used in leisure innovation if and only if, in addition, $f(M) = 0$.

The next proposition specializes this preference class to utility functions that are separable between consumption and leisure. This is a desired property for the study in this paper since M is interpreted throughout as leisure-technology.

Proposition 3. *A utility function that is separable between consumption and leisure, $u(c, g(h,M))$, is consistent with balanced growth in the decentralized equilibrium and in the optimal allocation if and only if it satisfies the relevant restrictions in Proposition 2 and $\varrho = 0$.*

Finally, the following corollary shows how the utility function of Section 2 fits in with the BGP preference class derived here.

Corollary 1. *The utility function in (9) belongs to the class in (15) with $\sigma = 1$, $\epsilon = 0$, $\varrho = 0$, $v(x) = \exp(-x)$, $f(M) = M^{\zeta}$.*

Proposition 2 extends the work of King *et al.* (1988), Boppart and Krusell (2020) and Kopytov *et al.* (2020) to the environment with growth in zero-price leisure technology.[24] It characterizes the class of preferences for which simultaneous growth in wages and in

[23] For u to make economic sense and to have economically plausible implications, additional restrictions are of course required. For example, if $\varrho < 1$ utility is decreasing in hours worked only if v is a decreasing function. Assumption 1 below imposes additional restrictions on u.

[24] In particular, King *et al.* (1988) characterize the class of utility functions that are consistent with constant hours worked in presence of a steady growth in wages; their formulation obtains with $\epsilon = 1$ and $\varrho = \zeta = 0$. Boppart and Krusell (2020) point out that hours worked might decline on the BGP if the income effect of higher wages is larger than the substitution effect: this is the case when ϱ is positive. Kopytov *et al.* (2020) analyze the case with declining prices of recreational goods.

leisure technology are consistent with balanced growth at the macro level, both in the decentralized equilibrium and in the optimal allocation. As anticipated above, the BGP class in the planning problem is a subset of the class that delivers balanced growth in the equilibrium.

How do these preferences work? Leisure technologies in this setting *rotate* preferences in favor of leisure time and *shift* the level of utility upward, with parameters ζ and ϵ guiding the strength of the two effects.[25] To see the implications for hours worked along the BGP, note that the intratemporal optimality condition $u_c \cdot w = -u_h$ becomes:

$$(1-\epsilon)\frac{w(t)h(t)}{c(t)} = -\varepsilon_v\left(h(t)M(t)^{\zeta}\right) \tag{16}$$

where $\varepsilon_v(.)$ is the elasticity of the v function. The left-hand side of (16) must be constant on a BGP. As long as v is not isoelastic, the product hM^{ζ} must be constant too, implying that h decreases at a rate $-\zeta\gamma_M$ on a BGP. Thus, as long as ζ is positive and v is not isoelastic, leisure time and leisure technology are complements and hours decline at a constant rate as leisure technology improves.[26]

Proposition 3 specializes these classes of preferences to feature a separability between consumption and leisure, in the usual sense that the marginal rate of substitution between leisure time and leisure technology is independent of consumption. It turns out that the only additional restriction required is that $\varrho = 0$, i.e. that the income and substitution effects of rising wages exactly offset.

Corollary 1 shows that the utility function (9) fits in the class that guarantees the existence of BGP in equilibrium. Since $f(M) \neq 0$, this utility function is *not* consistent with balanced growth in the optimal allocation with endogenous M.[27]

The propositions state the necessary and sufficient conditions for a utility function to be consistent with balanced growth asymptotically as represented by an interior solution to the relevant utility maximization problem. Additional restrictions must be imposed so that the first order conditions are indeed sufficient, and that the utility function makes economic sense. In the remainder of the paper I assume the following:

[25] The third parameter, σ, is the intertemporal elasticity of substitution.

[26] The remaining assumptions imply additional restrictions on v. For example, a utility function that belongs to the assumed class is increasing in M if the elasticity of v satisfies $-\varepsilon_v < \frac{\epsilon}{\zeta}$. The elasticity takes as an argument the product hM^{ζ}, it is thus endogenous, and so this condition must be verified in equilibrium. As we shall see in Proposition 5, the value of the elasticity in the decentralized equilibrium that I study in the next section is such that this condition is satisfied as long as $\frac{\epsilon}{1-\epsilon} > \zeta(1-\alpha+\alpha^2\chi)$. Thus, ϵ must be sufficiently high, and I assume this is the case. The equivalent condition in the socially optimal allocation is $\frac{\epsilon}{1-\epsilon} > \zeta(1-\alpha)$ which is weaker and thus it holds by implication.

[27] Intuitively, in this case the marginal utility of leisure products u_M depends on leisure hours, which makes it impossible for (172) to hold along the growth path.

Assumption 1: Utility function u belongs to the class in Proposition 3 with $f(M) = 0$. It is increasing in c and in M and decreasing in h and is strictly quasi-concave, with $0 \leq \epsilon < 1$ and $0 \leq \zeta < 1$. Moreover, the elasticity of function $v(\cdot)$, $\varepsilon_v(\cdot)$, is an invertible function.

Assumption 1 ensures that household preferences are well behaved and that the balanced growth path features per-capita consumption growth. The assumption that the elasticity of function v is invertible rules out v being isoelastic and leads to a unique solution to the planner's problem analyzed below.

The remainder of the analysis is carried out for any utility function that satisfies Assumption 1. To recap, any such utility function takes the form

$$u(c,h,M) = \begin{cases} \frac{\left(c^{1-\epsilon}M^{\epsilon}v\left(hM^{\zeta}\right)\right)^{1-\sigma}}{1-\sigma} & \sigma \neq 1 \\ \log\left(c^{1-\epsilon}M^{\epsilon}\right) + \log v(hM^{\zeta}) & \sigma = 1 \end{cases}. \tag{17}$$

It is worth clarifying that all the analysis of the decentralized equilibrium with preferences given by (17) carries through to the more general case of preferences with $f(M) \neq 0$, and in particular to preferences assumed in Section 2. The focus on preferences in (17) is motivated by the comparison with the social planner's solution, which is feasible when the planning problem admits a balanced growth path.

Armed with this analysis and assumptions on preferences, I now turn to study in more details the origins and implications of leisure technologies.

4 A static model of endogenous leisure technology

Why does the market generate zero price technologies designed to capture consumers' time, attention and data? The theory developed in this section provides an answer to this question by formalizing a link between leisure technology and intangible capital in the form of brand equity.

A static economy is populated by a measure N of individuals with a utility function $u(c,h,M)$ in the class in (17) and is endowed with a given stock of technological advancement A and capital stock K. There are two sectors that employ workers: one produces traditional consumption goods; the other produces non-rival leisure goods. An *allocation* in this economy is a tuple $\{s_M, h\}$ consisting of the share of labor employed in the leisure sector s_M and hours worked by each worker h (with $\ell = 1 - h$ the leisure time of each individual).

In addition, a by-product of households' leisure time is brand equity capital B. The idea is that households who engage in leisure activities are exposed to advertisements, and this increases the familiarity and perceived attractiveness of the advertised products. The amount of brand equity generated depends linearly on leisure time, so that in the aggregate the economy generates up to $N\ell$ (total leisure time) of brand equity. Brand equity plays a central role in equilibrium, as it allows for indirect monetization of leisure products. It is however essentially irrelevant in a planner's problem, because the planner decides directly on the size of each sector, and because she recognizes that the effects of the use of brand equity wash out in the aggregate (in the language of the IO literature, brand equity competition is combative – I discuss this in more detail below).

Specifically, the technologies used to produce the consumption good are the same as in Section 2 (equation (7)) except now the relative brand equity investments act as a demand shifter for each variety:

$$Y = \int_0^A \left(\left(\frac{b_i}{\bar{b}} \right)^{\chi\Omega} x_i \right)^{\alpha} L_Y^{1-\alpha} di \tag{18}$$

where $L_Y := (1 - s_M)hN$ is total labor hours devoted to the production of the traditional good, and the variable $\bar{b} := \frac{1}{A}\int_0^A b_i di$ is the average brand equity across all firms, so that the fraction $\frac{b_i}{\bar{b}}$ is firm i's brand equity relative to its competitors.[28] Parameter $0 \leq \chi < \frac{1-\alpha}{\alpha}$ measures the effectiveness of brand equity as perceived by the intermediate producers.[29]

4.1 Optimal allocation

It is immediate that the optimal allocation features symmetry in the output and brand equity investments across intermediate varieties i. Thus $b_i = \bar{b}\forall i$ and $x_i = \frac{K}{A}\forall i$ so that (18) collapses to $Y = K^{\alpha}(AL_Y)^{1-\alpha}$. Given K, A and N, the planner then chooses hours worked h, the share of labor employed in the leisure sector s_M, and brand equity B to maximize the utility of a representative individual:

$$\max_{s_M,h,B} u(c, h, M) \tag{19}$$

[28]The final-good firms anticipate any shifts in relative demand due to firms' intangible capital investments and demands more of the varieties with higher brand equity. This setup is isomorphic to the model where consumers were choosing the products directly and their relative taste for specific varieties was driven by brand equity.

[29]Ω is an indicator variable which equals to 1 when $\bar{b} > 0$ and 0 when $\bar{b} = 0$, making (18) well-defined in the case when there is no brand equity. The upper limit on χ is dictated by the requirement that intermediate producers make non-negative profits in equilibrium.

subject to

$$Y = K^\alpha (AL_Y)^{1-\alpha}, \tag{20}$$

$$c = Y/N, \tag{21}$$

$$M = L_M, \tag{22}$$

$$\ell = 1 - h, \tag{23}$$

$$B \leq N\ell. \tag{24}$$

The planner faces five constraints. Equation (20) is the consumption good production technology derived above. Equation (21) is the consumption good resource constraint: the consumption good is rival, so that each individual can consume an N-th share of whatever is produced. Equation (22) is the leisure production technology; I assume, without loss of generality, that leisure products are produced with labor, $L_M := s_M hN$. Importantly, non-rivalry means there is no equivalent to (21) here; instead, the aggregate M enters directly in utility of every individual. Leisure products are disembodied, and the usage by one individual does not preclude usage by another. Consequently, I use the terminology "technologies", "products" and "goods" interchangeably.[30] Equation (23) is the time endowment constraint. Finally, (24) reflects the generation of brand equity.

Since the brand equity B does not feature in the objective or in any of the other constraints beyond (24), we have the following result:

Lemma 1. *The planner is indifferent between producing any amount of brand equity between 0 and $N\ell^{SP}$ where ℓ^{SP} is the optimal leisure hours. If the real resource cost of production of brand equity was positive then the planner would choose to produce none.*

This result will also be useful in interpreting the equilibrium setting below: even though brand equity competition is a rat-race, non-zero brand equity does not in and of itself indicate an inefficiency. Given this result, we can solve for the socially optimal allocation by ignoring the final constraint. An interior optimal allocation satisfies the following first-order conditions:

$$u_c \frac{1}{N} \frac{\partial Y}{\partial h} + u_M \frac{\partial M}{\partial h} = -u_h \tag{25}$$

$$N \frac{u_M}{u_c} = -\frac{\frac{\partial Y}{\partial s_M}}{\frac{\partial M}{\partial s_M}}. \tag{26}$$

[30]Modern growth theory highlights non-rivalry as a critical feature of ideas (e.g. a drug recipe can be used over and over). Yet the products based on these ideas are rival (e.g. a box of pills). Non-rivalry in the case of leisure technologies is stronger, in the sense that not only ideas, but also the products themselves, are non-rival.

Condition (25) states that at the optimum the marginal benefit of increasing hours worked – the utility value of extra (per capita) output and leisure varieties – is equal to the marginal cost, which is the marginal disutility of work. Condition (26) states that at the optimum there can be no utility gain from shifting labor across sectors: the sum of MRSs across the population is equal to the marginal rate of transformation. This is reminiscent of the classic result on the optimal provision public goods by Samuelson (1954). This connection is due to the non-rivalry of leisure products, which makes them similar to public goods.[31]

Given the technologies and the utility class, optimality conditions (25) and (26) yield the following result:

Proposition 4. *There exists $\bar{N}$ such that, if $N \geq \bar{N}$, the socially optimal share of labor employed in leisure-R&D is*

$$s_M^{SP} = \frac{\frac{\epsilon}{1-\epsilon} - (1-\alpha)\zeta}{\frac{\epsilon}{1-\epsilon} + (1-\alpha)} \tag{27}$$

and hours worked are:

$$h^{SP} = \left[\frac{\Delta}{s_M^{SP} N}\right]^{\frac{\zeta}{1+\zeta}}, \tag{28}$$

where Δ is a parameter, derived in the Appendix, that is independent of s_M^{SP} and N and depends on the functional form of v.

Otherwise, $h^{SP} = 1$ and s_M^{SP} solves

$$\frac{s_M}{1-s_M} = \frac{1}{1-\alpha} \frac{\epsilon + \zeta \varepsilon_v \left((s_M N)^{\zeta} \right)}{1-\epsilon}.$$

Thus, if $N < \bar{N}$, $h^{SP} = 1$ and s_M^{SP} varies with the size of the economy (with N).

The optimal share of resources devoted to the production of leisure products (equation (27)) depends on the parameters of the utility function. This is not unexpected – naturally, the planner allocates a greater share of resources to production of leisure goods if those are highly valued by households. The share can be pinned down explicitly without the knowledge of function v. The specific form of v plays a role only in determining the optimal time allocation, through Δ in equation (28).

[31] Leisure products are not public goods because they are in principle excludable: individuals can be denied access to a leisure technology (for example, in a brief period following its inception, access to Facebook was restricted to a small network of universities in the US and in the UK). However, as we shall see below, in the equilibrium considered below no firm finds it optimal to exclude any consumers.

4.2 Decentralized equilibrium: attention economy

I now turn to an equilibrium theory of leisure technologies. The economy faces the same technological constraints as in the planner's problem above, but it is now the decisions of individual actors and interactions among them that determine the allocations.

4.2.1 Traditional production and brand equity competition

Competitive final good producers have access to technology in (18). They hire labor L_Y at wage w and combine it with differentiated intermediate goods $x_i,\ i \in [0, A]$ purchased at prices p_i. Assuming momentarily that $\bar{b} > 0$, the final good producers solve:

$$\max_{x_i, L_Y} \int_0^A \left(\left(\frac{b_i}{\bar{b}} \right)^{\chi} x_i \right)^{\alpha} L_Y^{1-\alpha} di - \int_0^A x_i p_i di - w_Y L_Y, \tag{29}$$

The first-order condition with respect to x_i gives the following conditional demand function:

$$\alpha \left(\frac{b_i}{\bar{b}} \right)^{\alpha\chi} x_i^{\alpha-1} L_Y^{1-\alpha} = p_i. \tag{30}$$

Equation (30) shows that a firm can shift out the demand for its product if it invests more in brand equity than its competitors: brand equity is all about *relative* advantage.[32] Just like in the planner's problem, in a symmetric equilibrium the $\frac{b_i}{\bar{b}}$ term vanishes and thus brand equity investments have no direct impact on aggregate productivity or consumer welfare. This formulation thus abstracts from the *direct* channels through which the *use* of brand equity may have macroeconomic effects, in either direction.[33] [34]

[32] This formulation is based on the empirical evidence on advertising, starting from the early studies such as Borden (1942) and Lambin (1976), and through more recent work summarized in Bagwell (2007). This literature suggests that brand equity and marketing activities may have a positive (although usually short-lived) impact on individual firm's sales, but that the effects tend to disappear once the unit of observation is expanded to a sector-level.

[33] The literature has distinguished three broad views of advertising: the *persuasive view* which sees advertising as primarily shifting demand curves outwards or lowering the elasticities of substitution across goods; the *informative view* according to which ads help consumers make better choices; and the *complement view* which sees ads as complements to the advertised consumption goods. The formulation in this paper is most closely aligned with the persuasive view with an additional assumption that direct effects wash out in equilibrium.

[34] On the "positive" side, brand equity investments may provide consumers with useful information about available products, which might lead to fiercer competition, lowering the distortion that arises from market power (Nelson, 1974; Butters, 1977; Grossman and Shapiro, 1984; Milgrom and Roberts, 1986; Stahl, 1989; Rauch, 2013); they may complement consumption goods (Becker and Murphy, 1993); or, when interpreted as accumulation of information and data, they can help firms better target consumer needs (Farboodi and Veldkamp, 2019). But brand equity competition may also lead to greater product differentiation, raising markups and exacerbating market power distortions (Molinari and Turino, 2009);

One concern about this formulation might be that the rat-race assumption automatically generates an inefficiency of the equilibrium. However, recalling Lemma 1, this is not the case: the planner does not care about the amount of brand equity per se. The reason is that the rate-race assumption is effectively neutralized by the fact that brand equity only requires leisure time to be produced. As a result, formulation in (18) is neutral also from the normative standpoint, in the sense that it does not, by itself, result in an inefficiency.

Consider now the problem of intermediate firms. Each producer is endowed with a single blueprint and produces its differentiated product using a technology that is one-to-one in capital, which it rents at cost r. Moreover, she can advertise her product by purchasing brand equity at price p_B. The resulting profit maximization problem is simply

$$\max_{k_i, b_i} \; p_i k_i - r k_i - p_B b_i \tag{31}$$

subject to the demand curve in (30), and taking r, p_B, L_Y and $\bar{b}$ as given. Solution to this problem then implies that the aggregate output can be written as $Y = K^{\alpha}(AL_Y)^{1-\alpha}$ and the price of brand equity is

$$p_B = \alpha^2 \chi \frac{Y}{B}. \tag{32}$$

4.2.2 Brand equity production and leisure R&D

Production of leisure goods (which I also call "leisure R&D") and brand equity is the domain of the platforms – two-sided businesses that form the core of the attention economy. I now lay out the technologies these firms use, discuss the market structure, and specify their decision problems.

4.2.3 Platform technology

Each platform is endowed with two technologies, corresponding to (22) and (24) in the planner's problem:

$$M_j = L_{M,j} \tag{33}$$

$$B_j = \ell_j, \tag{34}$$

aggressive advertising might become a nuisance to consumers (Johnson, 2013); data collection might raise privacy concerns (Tucker, 2012; Jones and Tonetti, 2020); or perhaps advertising may lead to envy or support 'conspicuous consumption', ultimately diminishing consumers' utility (Veblen, 1899; Benhabib and Bisin, 2002; Michel *et al.*, 2019). The formulation in (18) puts all of these considerations aside. In that sense it is *neutral*, and allows the theory to focus on the macroeconomic consequences of how brand equity is produced, rather than how it is used.Appendix J considers two non-neutral ways of modeling brand equity competition.

where $L_{M,j}$ denotes labor employed by platform j and ℓ_j is the share of leisure time captured by platform j. If in equilibrium consumers divide their time uniformly across activities, this share is proportional to the relative supply of leisure technologies of platform j, $\frac{M_j}{M}$. In principle, platforms can exclude some of the households from their services, so that

$$\ell_j = N_j \frac{M_j}{M} \ell, \tag{35}$$

where $N_j \in [0, N]$ is the choice of platform j (and denotes the measure of households that have access to its leisure technologies).

4.2.4 Market structure

There is free entry into the platform sector: there is a large mass potential entrants that can each hire an infinitesimal unit of labor to create leisure technologies. However, the existing leisure varieties are easy to copy and there is a lack of enforcement of property rights. Specifically, there is a competitive fringe that can supply consumers with leisure products (within leisure varieties that have already been created) at marginal cost, which is zero. This implies that platforms cannot charge positive prices for leisure varieties they invent (and I assume that transaction costs prohibit negative prices). But platforms can nonetheless break even, because of the revenues from brand equity sales. I assume that the fringe does not have access to the brand equity production technology.[35]

4.2.5 Platform's problem

Platform j solves the following problem:

$$\max_{L_{M,j}, N_j} p_B \cdot B_j - w \cdot L_{M,j} \tag{36}$$

subject to (33), (34) and (35), taking p_B, ℓ, and M as given.

4.2.6 Households

Individuals choose how much labor to supply at the ongoing wage, taking all variables, including the range of leisure technologies M, as given. The problem they solve is

$$\max_{h} u(c, h, M) \text{ s.t. } c = wh + rk + \tilde{\pi} \tag{37}$$

[35] This final assumption underscores the fact that, while there is marginal cost pricing of leisure products in equilibrium, the nexus between ideas, non-rivalry, increasing returns and imperfect competition continues to play an important role in this framework. An alternative market structure is to assume that platforms engage in Cournot competition in the brand equity market. This possibility is outlined in Appendix F.

where rk is the per capita capital income and $\tilde{\pi} := \frac{A\bar{\pi}}{N}$ are the intermediate producers' profits expressed in per-capita terms. The solution satisfies

$$-\frac{u_h}{u_c} = w. \tag{38}$$

I assume that implicitly households allocate their leisure time symmetrically across leisure activities.[36]

4.2.7 Market clearing and equilibrium

In equilibrium the share of labor employed in the leisure sector $s_M := \frac{L_M}{hN}$ is such that wages are equal across sectors. Market clearing conditions for goods-, labor-, capital- and brand equity markets are:

$$Y = cN, \tag{39}$$

$$L_Y + L_M = hN, \tag{40}$$

$$\int k_i di = K, \tag{41}$$

$$B := \int B_j dj = \int b_i di. \tag{42}$$

Definition 1. Given capital stock K and a level of traditional technology A, a *static equilibrium* is the pair $\{h, s_M\}$, and sets of firm-level quantities $\{x_i, b_i\}$ and prices $\{p_i, p_B, w, r\}$ and platform activity indicator Ω such that: households solve (37) and allocate leisure time uniformly across activities; final-good producers solve (29); intermediate producers solve (31); platforms solve (36); there is free entry to the leisure R&D sector; wages across sectors are equalized; market clearing conditions (39) — (42) hold; if no platforms are active, $s_M = B = \Omega = 0$ and M is equal to some small $\underline{M} > 0$ so that $u(c, h, M)$ is well defined. Otherwise, $\Omega = 1$ and $M = s_M hN$.

4.2.8 Characterization

Given the zero marginal cost of serving each user, no platforms restrict access to leisure varieties, and so $N_j = N \forall j$. Since wages are equal in both sectors, workers enter the leisure sector until the free entry condition

$$p_B B = wL_M \tag{43}$$

is satisfied. Using (32) in (43) yields the following result:

[36] Section 2 showed how this is an explicit choice of households with preferences in (1).

Proposition 5. *There exists $\tilde{N}$ such that if $N > \tilde{N}$, equilibrium share of labor employed in leisure-R&D is*

$$s_M^{DC} = \frac{1}{1 + \frac{1-\alpha}{\alpha^2 \chi}} \tag{44}$$

and the equilibrium hours worked are

$$h^{DC} = \left(\frac{\tilde{\Delta}}{s_M^{DC} N} \right)^{\frac{\zeta}{1+\zeta}} \tag{45}$$

where $\tilde{\Delta}$ is a parameter, derived in the Appendix, that is independent of s_M^{DC} and N and depends on the functional form of v.

If $N \leq \tilde{N}$, $s_M^{DC} = 0$ and $h^{DC} = 1$.

Equilibrium share of labor in the M-producing sector in (44) increases in the perceived effectiveness of advertising χ and in the intermediates' production share α, but, strikingly, is completely independent from household preference parameters. These comparative statics illustrate that in equilibrium the level of leisure technology is determined by what happens in the brand equity market: higher χ or α increase the demand for brand equity; to satisfy that demand, platforms employ more workers and invent more leisure varieties. But these decisions are not affected by how much value the leisure technologies bring to the consumers.

For a sufficiently small size of the economy (indexed by N), the hypothetical supply of leisure technology in equilibrium is so low that households optimally choose a corner solution of $h = 1$. This prohibits brand equity production, so that no workers are employed in the leisure sector. As we shall see, in the dynamic model this mechanism will mean that the leisure sector emerges only once the economy reaches a certain size.

4.3 Discussion of the inefficiency

This section considers the inefficiency when N is sufficiently large, greater than the thresholds $\bar{N}$ and $\tilde{N}$, so that both the socially optimal and the equilibrium allocations feature $h < 1$. A comparison of (27) and (44) shows that the market equilibrium does not allocate resources efficiently across sectors. Given this wedge, the time allocation decisions pinned down by equations (28) and (45) also differ across the two allocations. The underlying reason for the inefficiency is the lack of property rights – the fringe prevents the platforms from monetizing their leisure ideas directly. Given the extreme form of increasing returns to scale – inventing a new variety is costly but the marginal cost of supplying additional customers is zero – the lack of property rights would normally preclude the existence of an equilibrium with positive supply of M. The indirect monetization means that such equilibrium can exist, as long as the platforms can recoup their costs on the brand equity side of the market (i.e. as long as the fringe does not have access to the brand equity technology).

The explicit formulas for the share of labor employed in the leisure sector in the social optimum and in the decentralized equilibrium allow for a clean comparison of the two allocations:

Proposition 6. *There is over-provision of leisure products in the decentralized equilibrium,* $s_M^{DC} > s_M^{SP}$, *if*

$$\underline{\xi} < \frac{\epsilon}{1-\epsilon} < \underline{\xi} + \alpha^2\chi, \tag{46}$$

where $\underline{\xi} := \zeta(1-\alpha+\alpha^2\chi)$, *and under-provision,* $s_M^{DC} < s_M^{SP}$, *if*

$$\frac{\epsilon}{1-\epsilon} > \underline{\xi} + \alpha^2\chi. \tag{47}$$

To interpret the result in the proposition, note that the assumption that utility is increasing in M puts a lower bound on the value of ϵ, and, by extension, on $\frac{\epsilon}{1-\epsilon}$ (see also footnote 26). In the notation of Proposition 6, $\underline{\xi}$ denotes this lower bound (for the fraction $\frac{\epsilon}{1-\epsilon}$). The proposition then says that there is a sliver of parameter space of width $\alpha^2\chi$ where s_M^{DC} is above s_M^{SP}. The direction of inefficiency is thus ambiguous in sign. This is intuitive: whether the market allocates too much or too little labor to the leisure sector in general depends on the strength of the demand for brand equity (which determines the allocation in the equilibrium) relative to the utility value it brings to the consumers (which pins down the allocation in the planner's problem).

Examining the size of the two regions reveals that the region with oversupply is relatively small. To see this, note that Proposition 5 implies $\frac{wL_M}{Y} = \alpha^2\chi$: the size of the oversupply region in Proposition 6 happens to be the same as the model-implied size of the leisure sector measured by its cost share in output. A benchmark calibration target for the size of the leisure sector that is consistent with the data in the first panel of Figure 1 is under 1%. The interval in (46) is thus very narrow. To underscore this point, Figure 2 plots the two shares across a range of values for ϵ and ζ (in each panel) and χ (across panels). Each panel considers only the values of parameters that imply that utility increases with M.[37] All four panels show that the size of the oversupply region is small; but this is especially true for the two top panels showing low-χ calibrations, which are the calibrations that deliver a realistic size of the leisure sector (for standard values of

[37] Since s_M^{DC} does not depend on the preference parameters and since it increases in χ, it is a flat surface in all four panels with the level that rises with χ. s_M^{SP} does not depend on χ so it is identical in all four panels; it is decreasing in ζ and increasing inϵ.

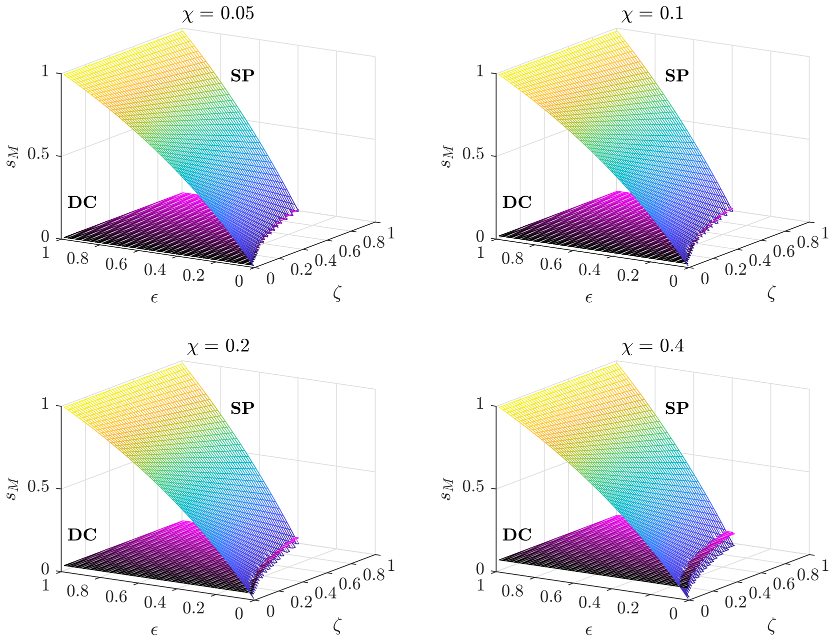

Figure 2

The share of labor in leisure R&D in equilibrium and in the optimal allocation

Notes: the Figure assumes that the size of the economy is sufficiently large so that hours worked are interior.

the capital share).[38] The main takeaway from the Figure is that only for very low values of ϵ, at which consumers are essentially indifferent about leisure technologies, we have that $s_M^{SP} < s_M^{DC}$. To the extent that the free leisure goods are highly valued by users (as suggested by the literature, see e.g. Brynjolfsson *et al.* (2019)), ϵ is well above the lower bound of $\underline{\xi}$ and indirect monetization results in undersupply.[39]

4.4 Direct monetization can restore efficiency

Consider an alternative decentralized economy with three changes relative to the equilibrium defined in Definition 1: (i) technology that produces brand equity (34) is not available, or alternatively $\chi = 0$: either way, platforms are no longer able to monetize consumers' leisure time through sales of brand equity; (ii) there continues to be free entry to the platform sector, but the property rights are protected: there is no competitive fringe that can supply users with existing leisure varieties at the marginal cost of zero; (iii) there is no market power in the intermediates-producing sector. Now, to cover the upfront costs of producing non-rival leisure technologies platforms charge a positive

[38] The parametrization in Section 6 sets $\chi = 0.07$.

[39] One important caveat to this conclusion is that in an open economy setting a leisure product might be used in many countries. This would raise the private returns to leisure innovation and thus direct more resources towards that sector.

price.[40] Denoting this price with p_M, free entry into leisure-R&D implies:

$$p_M M \cdot N = w \cdot L_M. \tag{48}$$

Households solve

$$\max_{h,M} u(c,h,M) \text{ s.t. } c + p_M M = wh + rk$$

which yields the usual optimality conditions $w = -\frac{u_h}{u_c}$ and $p_M = \frac{u_M}{u_c}$. Combining these with the budget constraint and noting that competitive behavior and market clearing imply $rk = \alpha y = \alpha c$, we obtain equation (28). s_M is determined from the equality of wages across the two sectors; the solution gives (27). Thus an equilibrium with property rights and inability to indirectly monetize the leisure products yields an efficient outcome. I summarize these results in the next proposition:

Proposition 7. *The static decentralized equilibrium is efficient in an economy in which platforms charge subscription fees for the leisure varieties they produce. This is the case if there is no cross-subsidization through brand equity, if property rights are protected so no competitors are able to charge the marginal cost of zero, and if the intermediate producers behave competitively, e.g. because there is free entry.*

4.5 Taking stock

The dynamic model of Section 2 introduced the concept of leisure varieties within a simple activity-based leisure framework and showed steady progress in leisure technology can result in a downward trend in hours worked and lower growth of traditional TFP. Section 3 characterized the entire class of utility functions that deliver these insights in a balanced growth framework. And Section 4 developed a theory of how zero-price leisure technologies arise in a decentralized equilibrium (they capture consumers' time which is a critical input into the generation of intangible capital) and analyzed the efficiency properties of such equilibrium. The next section combines these insights in a full dynamic model, teasing out the implications for the process of economic growth.

[40]Note that even if there is a subsidy to the production of intermediate products correcting for the market power distortion, the equilibrium is still inefficient. That is, that the issue is really about the indirect monetization of leisure products, not the monopoly distortion for intermediate goods.

5 Growth with endogenous leisure technology

The full model features a dynamic economy consisting of three sectors: the traditional production sector, the traditional R&D sector, and the leisure R&D sector. The technology and resource constraints are summarized below:

$$Y(t) = \int_0^{A(t)} \left(\left(\frac{b_i(t)}{\bar{b}(t)} \right)^{\chi \cdot \Omega} x_i(t) \right)^{\alpha} L_Y(t)^{1-\alpha} di \tag{49}$$

$$x_i(t) = k_i(t), \quad \int_0^A k_i(t) di = K(t), \quad \int_0^A b_i(t) di = B(t) \tag{50}$$

$$\dot{A}(t) = L_A(t) A(t)^{\phi} \tag{51}$$

$$\dot{M}(t) = L_M(t) A(t)^{\phi} \tag{52}$$

$$B(t) = N(t)\ell(t) \tag{53}$$

$$\ell(t) = 1 - h(t) \tag{54}$$

$$N(t)h(t) = L_Y(t) + L_A(t) + L_M(t) \tag{55}$$

$$\dot{K}(t) = Y(t) - C(t) - \delta K(t). \tag{56}$$

Traditional production and R&D sectors. Final good producers are competitive and operate technology (49). As in Section 2, there is free entry to traditional R&D and new blueprints are invented according to technology (51). Each blueprint is sold to an intermediate producer at a price $V(t)$, granting the rights to produce a given variety in perpetuity using production technology in (50). Each intermediate firm solves the profit maximization problem in (31).[41]

Leisure R&D (platform) sector. The market structure is as in Section 4: there is free entry into leisure R&D, and a fringe that is able to copy the existing leisure varieties and sell them at the marginal cost of zero. For symmetry with traditional R&D, the leisure ideas production function is dynamic, so that leisure technology accumulates over time, and, just as in traditional R&D, the probability of success in leisure-R&D also depends on the stock of traditional knowledge A^{ϕ} (equation (52)). While such a spillover from traditional to leisure R&D is realistic, it is not necessary for any of the results. Indeed, the symmetry of the spillover term across the two R&D technologies is not important for the results. It has the implication that the two technologies grow at identical rates in

[41] This assumes that brand equity investments depreciate fully each period. The literature usually finds high depreciation rates on the order of 50% per annum; given the long-run focus of this paper the assumption of full depreciation of brand equity is appropriate.

steady state, but this is not a pre-requisite for a BGP.[42] More generally, the qualitative results that follow are robust to various ways of modeling the leisure R&D technology. For example, a static formulation in (22) could be used in the dynamic model; it is also straightforward to incorporate two-way technological spillovers between the two R&D sectors without changing the long-run results, or include other inputs, such as final good or capital.[43]

With the leisure ideas production function in (52), platform j's problem becomes dynamic. Each platform maximizes profits taking all aggregate variables as given. The same arguments as before imply that no platform restricts access to its technologies: $N_j(t) = N(t) \forall j$. Thus each firm solves the following problem:

$$\max_{L_{M,j}(t)} \int_0^\infty e^{-\int_0^t r(\tau)d\tau} \left(p_B(t) \cdot B_j(t) - w_M(t)L_{M,j}(t)\right) dt \quad \text{subject to} \tag{57}$$
$$B_j(t) = M_j(t)\frac{\ell(t)}{M(t)}$$
$$\dot{M}_j(t) = L_{M,j}(t)A(t)^\phi$$

taking all aggregate variables, including p_B, as given. The solution satisfies:

$$Z(t)A(t)^\phi = w_M(t) \tag{58}$$
$$p_B(t)\frac{\ell(t)}{M(t)} = r(t)Z(t) - \dot{Z}(t). \tag{59}$$

where $Z(t)$ is the costate variable in the optimal control problem of the firm – the shadow value of the leisure blueprint (the problem is laid out in Appendix B).

Households. Population $N(t)$ grows at rate n. A representative household solves the following problem:

$$\max_{\{c(t),h(t)\in[0,1]\}_0^\infty} \int_0^\infty e^{-\rho t}u(c(t), h(t), M(t))dt \quad \text{subject to} \tag{60}$$
$$\dot{K}(t) = w(t)h(t)N(t) + r(t)K(t) + A(t)\bar{\pi}(t) - \dot{A}(t)V(t) - c(t)N(t)$$

and the time endowment constraint (54), where $u(.)$ satisfies Assumption 1, $\dot{A}(t)V(t)$ denotes the spending on purchases of new blueprints and $A(t)\bar{\pi}(t)$ are the profits of the

[42] With $\dot{M} = L_M A^{\phi_M}$ and $\dot{A} = L_A A^{\phi_A}$ and $\phi_M \neq \phi_A$, the steady state growth rates of M and A will be different, but BGP will still exist and will have the same structure as in the symmetric $\phi_M = \phi_A$ case. See also footnote 46 below.

[43] In other words, none of the substantive results would change if we imposed a richer and more general leisure ideas production function such as $\dot{M} = L_M^\varphi X^{1-\varphi} \cdot (\lambda_A A^{\phi_A} + \lambda_M M^{\phi_M})$, where labor is combined with final good X and there are two-way technological spillovers across the R&D activities.

intermediate firms.

Capital depreciates at rate δ, so that the resource constraint of the economy is given by (56). Equations (50), (53), (55) and (56) are the market clearing conditions for the final good, capital, labor and brand equity markets.

5.1 Equilibrium

Definition 2. Given K_0, A_0 and N_0, a *dynamic equilibrium* is a set of paths of aggregate quantities $\{Y, C, K, A, M, B, h, \ell, s_A, s_M\}_{t=0}^{\infty}$; firm-level quantities $\{x_i(t), b_i(t)\}_{t=0}^{\infty}$; prices $\{p_i(t), p_B(t), V(t), Z(t), w(t), r(t)\}_{t=0}^{\infty}$ and platform activity indicator $\{\Omega\}_{t=0}^{\infty}$ such that: households solve (60) and allocate time uniformly across activities; final-good producers solve (29) at each point in time; intermediate producers solve (31) at each point in time; platforms solve (57); there is free entry to traditional- and leisure R&D; wages are equal in all sectors; markets for final good, capital, labor and brand equity clear; if no platforms are active, $s_M = B = \Omega = 0$ and and M is equal to $\underline{M} > 0$ so that $u(c, h, M)$ is well defined. Otherwise, $\Omega = 1$; if for all $t' \leq t$, $\Omega(t') = 0$ then $\mathbb{E}_t\Omega(t'') = 0$ for at any $t'' > t$. Otherwise agents have perfect foresight.

The definition mirrors Definition 1 in the static model of Section 4. The last part of the definition states that in the dynamic equilibrium, no agents anticipate entry into the platform sector if no platforms had ever existed. This assumption is made for convenience, but it is not important nor necessary. It does ensure that growth is exactly balanced when $B = \Omega = 0$, simplifying the exposition. The economic rationale for it is that it might be difficult for agents to anticipate the arrival of previously unobserved kind of technology.[44]

5.2 Balanced growth path

Recall from the discussion in Section 4 that for a low enough level of the leisure technology, households optimally choose hours worked equal to 1. Because of this property, the balanced growth path is "segmented":

Definition 3. A *segmented balanced growth path* (sBGP) is an equilibrium in which: (i) initially, per capita consumption, output and traditional technology all grow at a constant rate, the interest rate, the share of labor in traditional R&D are constant and the share of labor in leisure-R&D is zero ("segment 1"); (ii) as $t \to \infty$, C, Y, A and M grow at possibly distinct but constant rates, h decreases at a constant rate, the interest rate and the shares of labor in the three sectors are constant ("segment 2").

[44] Appendix L solves for a perfect foresight equilibrium in which agents anticipate the entry of platforms and shows that the anticipation effects are small.

I assume that initial levels of K_0, A_0, $\underline{M}$ and N_0 are such that the economy is initially on segment 1 of the sBGP: growth is balanced and $h = 1$ for all $t < \hat{t}$, where $\hat{t}$ is defined as the point when platforms become active.

5.3 Long-run growth effects of endogenous leisure technologies

How does the nature of growth change as a result of leisure-enhancing technologies? The following proposition characterizes the growth rates of the economy in segments 1 and 2 of the sBGP.

Proposition 8. *Let $\hat{t}$ denote the time when platforms become active. For $t \leq \hat{t}$ (segment 1) there is no leisure enhancing technological change, hours worked are constant and equal to 1, and per capita consumption, per capita output, wages and TFP all grow at the same constant rate, denoted γ_1, which is given by:*

$$\gamma_1 = \frac{n}{1-\phi}. \tag{61}$$

For $t \geq \hat{t}$, platforms are active and the economy transitions to segment 2 of the sBGP. Asymptotically, hours worked decline at a constant rate given by

$$\gamma_h = -\frac{\zeta n}{1-\phi+\zeta} \tag{62}$$

and the growth rates of traditional- and leisure technologies are:

$$\gamma_A = \gamma_M = \frac{n}{1-\phi+\zeta} < \gamma_1. \tag{63}$$

Per-capita output and consumption grow at:

$$\gamma_y = \gamma_c = \gamma_A \left(1-\zeta\right) < \gamma_1. \tag{64}$$

In segment 1 there are no platforms, $M = \underline{M}$ and $h = 1$. Growth is exactly balanced, given the assumptions about the starting levels of the state variables and lack of anticipation effects. These assumptions make segment 1 a convenient reference point: the expression in (61) is familiar from the canonical semi-endogenous growth model of Jones (1995).[45]

In the asymptotic segment 2, hours worked are no longer constant but are instead falling at a constant rate. The speed of this decline increases in ζ, which is the parameter

[45] The only difference is that I have implicitly assumed no R&D duplication externalities.

that indexes the degree of complementarity between leisure technology and leisure time in utility.

The emergence of leisure-enhancing technologies is associated with a decline in the long-run growth rate of traditional technology (equation (63)).[46] The mechanism is the same as the one that underlies Proposition 1: heightened competition for time and attention leaves less resources available for productive activities and reduces the effective market size for traditional technology. The declining hours worked and slower productivity growth both depress the growth rate of per-capita output and consumption (equation (64)).

5.4 Long-run allocative effects of leisure technologies

The emergence of the leisure-R&D sector reallocates resources across the economy. The next proposition pins down the equilibrium sectoral shares on the two segments of the sBGP.

Proposition 9. *For $t < \hat{t}$ (in segment 1 of the sBGP) the share of labor employed in the leisure R&D sector s_M is zero. The share of labor in the traditional R&D sector is:*

$$s_A = \frac{1}{1 + \frac{1}{\alpha}\frac{\rho+\gamma_1}{\gamma_1}} \tag{65}$$

This share is increasing in γ_1, defined in equation (61).

In segment 2 of the sBGP the share of labor employed in the traditional R&D sector converges to a constant given by

$$s_A = \left(1 - \frac{\alpha\chi}{1-\alpha}\right) \cdot \frac{1}{1 + \frac{1}{\alpha}\frac{\rho+\gamma_A}{\gamma_A}}. \tag{66}$$

Since $\chi > 0$ and $\gamma_A < g$, the share of labor in traditional R&D in segment 2 is lower than in segment 1. The share of labor employed in leisure R&D is:

$$s_M = \frac{\alpha\chi}{1-\alpha} \cdot \frac{1}{1 + \frac{1}{\alpha}\frac{\rho+\gamma_A}{\gamma_A}}. \tag{67}$$

Proof. Appendix B. ☐

The emergence of leisure technology leads to a redirection of innovative efforts: labor

[46] Along the sBGP leisure technologies grow at the same rate as traditional technologies. Alternative assumptions with regards to the shapes of the R&D technologies would result in different formulas but would not change the structure of the BGP. For example, it is easy to show that with $\dot{M} = L_M A^{\phi_M}$ $\dot{A} = L_A A^{\phi_A}$ BGP growth rates satisfy $\gamma_M = \frac{n(1-\phi_A+\phi_M)}{1-\phi_A+\zeta(1-\phi_A+\phi_M)}$ and $\gamma_A = \frac{n}{1-\phi_A+\zeta(1-\phi_A+\phi_M)}$.

shifts away from traditional R&D and towards leisure-R&D. The size of this reallocation is determined by $\alpha\chi$, which is closely related to the proportion of revenue that shifts away from the intermediate producers and towards the platforms as a result of brand equity purchases. The direction of profit-driven innovation follows the money.[47] Within the semi-endogenous growth framework these shifts generate (persistent) *level* effects, but do not affect the steady state growth rates in Proposition 8.

Proposition 10 in Appendix B derives a system of dynamic equations that fully characterize the equilibrium also outside of the steady state. I now turn to the optimal allocation, and derive a set of wedges that correspond to the inefficiencies of the market equilibrium.

5.5 Optimal dynamic allocation

The planner's problem is a straightforward dynamic extension of the static problem (19):

$$\max_{\{c(t):=\frac{C(t)}{N(t)}, s_A(t), s_M(t), h(t)\}_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t} u(c(t), h(t), M(t)) dt \quad \text{subject to}$$

$$\begin{aligned}
Y(t) &= K(t)^{\alpha} \left(A(t)(1 - s_A(t) - s_M(t))h(t)N(t)\right)^{1-\alpha} \\
\dot{K}(t) &= Y(t) - C(t) - \delta K(t) \\
\dot{A}(t) &= s_A(t)h(t)N(t)A(t)^{\phi} \\
\dot{M}(t) &= s_M(t)h(t)N(t)A(t)^{\phi} \\
B(t) &\leq N(t)\ell(t) \\
\ell(t) &= 1 - h(t) \\
s_A(t), s_M(t) &\in [0, 1] \\
\dot{N}(t)/N(t) &= n.
\end{aligned}$$

5.5.1 Optimal level of brand equity

Note first that Lemma 2 applies in the dynamic setting too: the planner internalizes the fact that brand equity competition is a wash and thus is socially useless. However, brand equity is also costless to generate (for a given quantity of leisure time), meaning that the

[47] The formulas in the proposition also show that, holding the growth rate of the economy fixed, the overall research share (traditional- plus leisure-R&D) is unchanged across the two segments. However, since per Proposition 8 the economy experiences a decline in growth, the model predicts the overall R&D share to decline. This prediction (which would be present in any semi-endogenous growth model with declining trend growth) is not borne out by the data, which shows the share of resources devoted to R&D increasing over the decades (Jones, 2015; Bloom *et al.*, 2020). Clearly, the model abstracts from many trends that could be behind this, such as globalization or improvements in IT technology. Olmstead-Rumsey (2022) estimates a model in which the R&D share increases as a result of a decline in the cost of R&D.

planner is indifferent as to how much brand equity to provide as long as this provision does not affect other allocations. It is thus without loss of generality to assume that $B^* = 0$.

5.5.2 Optimal growth along the BGP

The Appendix sets up a Hamiltonian and solves the optimal control problem corresponding to the planner's problem. By construction, for any utility function satisfying Assumption 1, the socially optimal allocation converges to a balanced growth path. Along the optimal path the asymptotic growth rates are the same as in the decentralized equilibrium. In particular, as $t \to \infty$, the optimal allocation converges to a BGP with growth rates of hours and technology given by

$$\gamma_h^* = -\zeta \gamma_A^* \quad \text{and} \quad \gamma_A^* = \gamma_M^* = \frac{n}{1 - \phi + \zeta}.$$

5.5.3 Optimal allocation, wedges and policies

A comparison of the equilibrium system (124)–(135) and the optimal allocation system (136)–(145) in the Appendix shows that there are three sources of inefficiency present in a dynamic market economy. First, there is the usual monopolistic distortion in the intermediate sector which shows up as a wedge between the equilibrium real interest rate r and the socially optimal return to capital. Second, because of knowledge spillovers and the shifting of profits from traditional to leisure firms, the social value of traditional R&D activity diverges from the private value. Third, the lack of property rights discussed in Section 4 means that the social value of leisure R&D activity is different from the private value. Appendix C specifies the set of tax-subsidy schedules that decentralizes the optimal allocation.

6 An illustrative quantification

The analytical results in the previous sections allow for a sharp characterization of the growth process in segment 1 and in the asymptotic segment 2. This section provides an illustrative quantification of the long-run effects and solves for the transitional dynamics between the two segments. The results in this section are illustrative and they do not represent a formal calibration exercise. Instead, the goal is to assess how macro-economically significant the effects highlighted above might be.

Parametrization corresponds to annual frequency, with the discount rate of 1% and population growth of 1% per annum. Several parameters are calibrated to standard values

Parameter	Description	Value	Target / source
ρ	Household discount rate	0.01	$r \approx 4\%$
n	Population growth	0.01	AEs data
α	Capital share	0.35	standard calibration
δ	Capital depreciation	0.05	standard calibration
ϕ	Returns to ideas in R&D	0.5	$\gamma_1 = 2\%$
χ	Perceived effectiveness of brand equity	0.07	empirical elasticities
ζ	Elasticity of hours to leisure technology	0.33	trend in hours

Table 1
Model calibration

– the capital share α equals 0.35, the depreciation rate δ is 5% per year. Parameter ϕ guides the degree of increasing returns to innovation and determines the steady state growth rate of the economy. Recent work by Bloom *et al.* (2017) has found that the ϕ parameter varies widely across sectors in the US economy, but is likely to be well below 1. I set it to 0.5, targeting the growth rate of the economy in segment 1 of 2%.[48] Parameter χ corresponds to the perceived effectiveness of brand equity. I set this parameter to 0.07. This value generates empirically plausible effects of brand equity purchases on sales (Bagwell (2007), DellaVigna and Gentzkow (2010), Lewis and Reiley (2014), Lewis and Rao (2015)) and matches the empirically plausible size of the leisure R&D sector in the aggregate.

In terms of the preference parameters, I assume utility function is the same as in Section 2: $u(c, h, M) = \log c + M^{\zeta}(1 - h)$, meaning that $\sigma = 1$, $\epsilon = 1$ and $f(M) = M^{\zeta}$. The only parameter left to calibrate is ζ. I set this parameter to 0.33, implying a which delivers a trend rate of decline in hours worked along the BGP that is equal to -0.4%, the long-run average rate of decline of hours across countries, and I explore how the results change for different parametrizations in Appendix K.

6.1 The magnitude of long-run growth effects

Plugging in the parameter values into the formulas in Propositions 8 and 9 shows that leisure technologies can have substantial macroeconomic effects. The growth of traditional technology falls from 2% to 1.2% along the sBGP.

[48] The numerical exercises here assume a static form of the leisure ideas production function $M = L_M A^{\phi}$ for simplicity.

6.2 Transition from segment 1 to segment 2 of the sBGP

In order to compare the simulated transition path with the observed trends one must first decide on the empirical counterpart to $\hat{t}$: the point at which platforms first become active. Inevitably, this requires some judgement. While examples of zero-price leisure varieties supplied by the private sector exist in the pre-war period, the mass roll-out of television, which in the case of the United States started in the late 1940s in the midst of the post-war boom, has revolutionized the world of mass-available leisure and advertising. Adoption along both the extensive and the intensive margins was rapid, with average time spent watching TV of 1.5 hours per day by mid-1960s (Figure A.3).[49] I thus assume that $\hat{t} = 1950$ and investigate the impact of leisure technologies on the post-WW2 growth patterns.

Figure 3 plots the transition path generated by the model against the trends observed in the data shown in Figure 1. The model matches the modest size of the leisure sector measured by Nakamura *et al.* (2017) as the cost of production of zero-price goods and services.[50] Despite this modest size, it has significant macroeconomic effects: the emergent leisure technologies lead to a substantial decline in hours worked and a slowdown in the growth rate of total factor productivity.[51] The emergence and steady growth of leisure technologies can account for around half – or 0.8 of a percentage point – of the slowdown in traditional TFP growth. Additionally, the top-left panel shows the growth rates of the number of radio and TV stations over time, against the model-implied path for the growth rate of M. These data are no doubt only imperfect proxies for the growth of leisure technology over time – not least, they do not capture the explosion of digital leisure products in the later part of the sample, nor do they account for the quality improvements in entertainment offerings. Nonetheless, this panel illustrates that the 1950s have seen a rise in leisure technologies that was spectacularly rapid, and that steady growth continued in subsequent decades. Both features are broadly consistent with what the model predicts.

[49] Adoption of the radio in the mid-1920s would be another candidate. While adoption of radio receivers occurred before World War 2, the top right panel in Figure 3 shows that the number of radio stations grew rapidly after the war, around the same time as television was being rolled out, lending some support to the choice of 1950 as the point of departure. Corroborating this judgement, Vandenbroucke (2009) finds that over the period 1900-1950 only about 7% of the shift in time allocation was due to leisure technology.

[50] In the baseline simulation, the size of the leisure sector is stable over time, but in the data we observe an increase that starts in the late 1970s and accelerates recently. The model suggests this increase may be a result of shocks within the leisure sector itself – for example, a steady increase in χ as well as improvements in platforms' productivity can match the rise in the size of the leisure R&D sector over time. These changes may perhaps be thought as crudely capturing developments such as the rise of network effects or user generated content. These results are available upon request.

[51] Section 7 shows that leisure technologies are not captured in the GDP statistics. Therefore the measured TFP growth shown in panel D corresponds to the growth rate of A in the model.

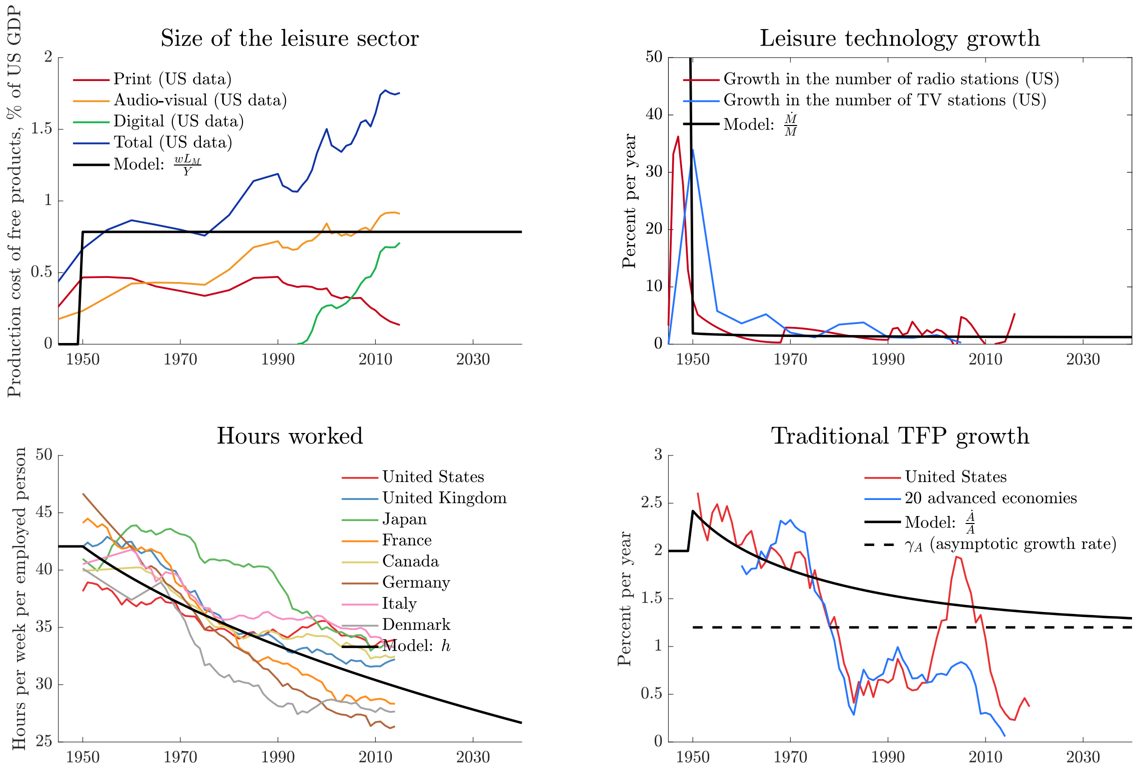

Figure 3

The Model's Growth Path versus the Trends Observed in the Data

Data sources as in Figure 1, except for the top-right panel in which the data are from the Federal Communications Commission and Statista. These data are interpolated over the missing values. The black lines with empty circles show the model's transition following the entry of platforms that is assumed to have taken place in 1950.

Two lessons emerge from this illustration. First, the effects of the emergence of the leisure sector can be quantitatively substantial: one should not discount the attention economy as an important explanation for macro trends merely because it is small as a share of GDP. Leisure goods are non-rival, meaning that they can be used over and over again by multiple people simultaneously, creating a possibility that a small sector can have outsized macroeconomic effects. Second, it is likely that the attention economy itself has undergone technological shifts over time. Put differently, while leisure technologies we see today represent, in part, a natural progression from those that we saw in the 1950s, there are also structural differences, perhaps related to developments such as emergence of data gathering and the rise of user-generated content. Measuring and formally modeling the nature and consequences of these shifts forms an exciting avenue for future inquiry.

7 Measuring the leisure economy

Leisure products generated within the economy are not captured in headline GDP statistics. The 2008 UN System of National Accounts views platforms of Sections 5 and 6 as advertising agencies: their output is brand equity B, which serves as intermediate inputs of the ad-buyers (Byrne *et al.* (2016b), Bean (2016)), while the zero price M products are not included. Two questions arise in this context. First, does this mean that GDP is significantly mismeasured? And second, do leisure technologies make GDP a less reliable guide to welfare over time? In this Section I explain why the answers to these questions are 'no' and 'yes', respectively.[52]

7.1 Production cost-based value of leisure technologies

GDP has been designed with the intention to measure market-based production.[53] Assuming this perspective, Nakamura *et al.* (2017) propose valuing leisure products with the cost of production, which is consistent with the usual treatment in the National Accounts. In the model economy, this measure of value is simply

$$V_1 := w \cdot L_M = \alpha^2 \chi Y. \tag{68}$$

The second equality follows from substituting in equilibrium wages. This measure of value of free services grows at the same rate as output. Moreover, the level effect is bounded by the size of this sector, which is relatively small.[54] Together, these observations mean that GDP is not *significantly* mismeasured, at least when there are no changes in the structural parameters such as χ or α that would cause a transitional upward shift in the size of the leisure sector.[55]

[52]Note that the M products are (i) complementary with leisure time; (ii) non-rival; (iii) monetized indirectly through capturing consumers' time and attention. Other products that interact with time – through complementing it (vacations) or being a substitute for it (meal delivery services) – are included in traditional consumption aggregator c and output Y.

[53]Nonetheless, in practice GDP does include elements that are outside of the production boundary, such as home production of goods or owner-occupied housing. Moreover, given the lack of an agreed comprehensive measure of economic wellbeing, it is often mis-used as a measure of welfare. See Jorgenson (2018) for an overview of the debate and Coyle (2017) for an extensive discussion of the production boundary in the context of digital goods.

[54]In the calibration discussed above, as a share of output, V_1 is under 1% of GDP – see panel A in Figure 3.

[55]See Moll *et al.* (2021) for analysis of the shifts in the capital share driven by automation.

7.2 Use-based measures of value

However, to capture the utility consumers obtain from the zero price leisure technologies, one must turn to measures that focus on the use of these technologies. One such metric – independent of the formulation of preferences – is the time spent with these services valued at an ongoing wage (Goolsbee and Klenow (2006), Brynjolfsson and Oh (2012)). Another is a model-consistent measure of welfare: a change in consumption required to compensate consumers for no access to leisure technologies. Denoting these with V_{2a} and V_{2b} respectively, we have:

$$V_{2a} := N \cdot w \cdot \int_0^M \ell(\iota) d\iota = \Phi \frac{1-h}{h} C \tag{69}$$

$$V_{2b} := N(\bar{c} - c) = \left(\left(\frac{M}{c} \right)^{\epsilon} v(hM^{\zeta}) \hat{f}(M) - 1 \right) C \tag{70}$$

where $\{\bar{c}\}_{t=0}^{\infty}$ in (70) solves $u(\bar{c}, 1, 0) = u(c, h, M)$ and $\{c, h, M\}_{t=0}^{\infty}$ are equilibrium paths of these variables and where $\hat{f}(M) := \exp(f(M))$.

Since h is declining and $1-h$ is increasing on a balanced growth path, the wage-based measure V_{2a} in (69) grows faster than aggregate output and consumption on a BGP. Its growth rate converges to a constant $\gamma_Y - \gamma_h$ asymptotically, as the growth rate of leisure time becomes negligible. The compensating variation measure V_{2b} in (70) is closely tied to preferences, with the growth rate that hinges on the function $f(M)$. In the benchmark BGP case with $f(M) = 0$ (and $\hat{f}(M) = 1$) the growth rate of V_{2b} converges to a constant $\gamma_Y - \epsilon\gamma_h$. In the case with $f(M) = M^{\zeta}$, which is implicit in the utility function in (9), the growth rate of V_{2b} increases over time.

The underlying reason for the difference between the production and use-based approaches is the strong non-rivalry: in the attention economy, the use of leisure technologies is detached from their production cost.

8 Conclusion

This paper formalized the idea of leisure-enhancing technologies: products that complement leisure time, are non-rival, and are monetized indirectly through capturing consumers' time and attention. The main takeaway is that zero price leisure technologies are a flip side of brand equity competition among firms, and that leisure innovation can have significant long-run effects on the macroeconomy. The paper has demonstrated that leisure technologies differ in economically significant ways from other innovations. Thus, they merit further attention and research. Two aspects are particularly worthy of further exploration.An open question is how the rise of the leisure sector interacts with household heterogeneity. Intuitively, zero price leisure technologies might matter most for the poorer households. These technologies might be important for understanding the rise in leisure inequality (Boppart and Ngai, 2017b), and for assessing how inequality of income translates to inequality of welfare. Furthermore, leisure technologies tend to diffuse rapidly across the world. This suggests that the market that guides the supply of leisure technologies becomes increasingly global, and it means that adoption of leisure technologies in emerging economies can be rapid even at low levels of output per capita. Such "premature adoption" would have interesting implications for growth and development prospects in these countries. Studies of leisure technology in an open-economy or global setting could shed light on these issues and help design policies fit for the world with a leisure technology sector.

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Appendices (for online publication only)

A Illustrative evidence

This Appendix further motivates the focus of this paper and forms a background to the analysis.

Evidence on leisure-enhancing innovations

Figure 1 in the main text illustrated the increased importance of the digital sub-sector of the attention economy since the mid-1990s. The available industry statistics reinforce this message. For example, Figure A.1 shows the dramatic rise in the number of smartphone apps, with the majority available free of charge to the consumer. The fact that millions of apps have been created over the past decade is a testament to the innovative efforts of firms in the attention economy.[56]

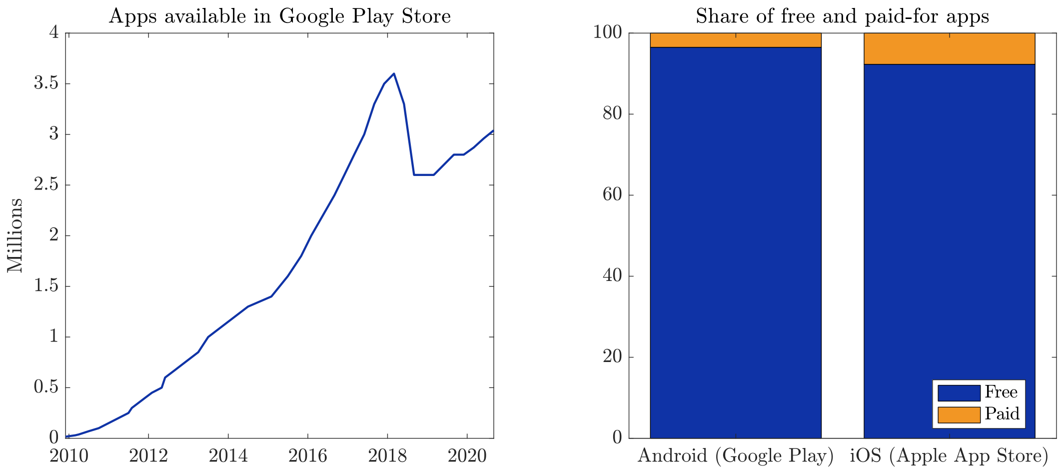

Figure A.1
Smartphone Apps

Source: The number of apps in Google Play Store is from Google, App Annie and AppBrain. The paid vs free apps breakdown is from 42matters, an app analytics company.

Consistent with the rapid technological progress within it, the leisure sector appears

[56]The market structure in the app market is more complex than in the model presented in the main text. Apps are available on platforms such as Google Play Store or Apple App Store, but are produced by many firms, not just Google or Apple. This additional layer of intermediation does not change the economics of the paper though: the incentives to capture the time and attention of the end-user remains. Future work could usefully explore the competition, business stealing and firm dynamics aspects of the app producers or other firms within the broadly defined leisure sector.

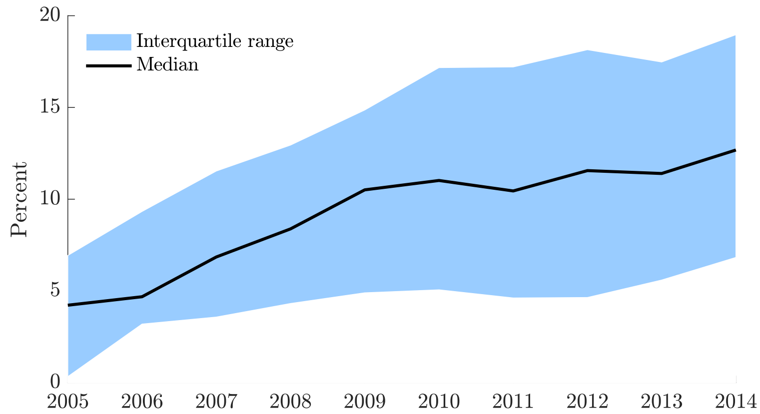

Figure A.2

R&D Expenditure Share of the (Proxy for the) Leisure Economy

Source: OECD. Includes data for an unbalanced panel of 39 countries. The figure shows the median and the interquartile range of the country-level share of R&D spending in the following sectors: publishing; motion picture, video and television program production; sound recording; programming and broadcasting activities; telecommunications services; computer programming, consultancy and related activities; information service activities; data processing, hosting and related activities; and web portals.

to be an increasingly important driver of the overall R&D spending. No exact measure for the share of attention economy in overall R&D spending is available; but it is possible to construct rough proxies by considering a subset of industries which are most likely engaged in leisure-enhancing innovations. Figure A.2 shows that the share of R&D spending accounted for by the sectors such as video and TV program production, sound recording, broadcasting and web portals has been rising over time.

Changes in time allocation patterns

The numerical simulation of the transition between segments 1 and 2 in Section 6 uses 1950 as the point of the emergence of leisure technologies. To support this, Figure A.3 illustrates the rapid adoption of television both along the intensive and extensive margins.

While hours worked in the United States have fallen by less than in other countries (recall the middle panel of Figure 1), the trend in leisure time has been clearly upwards. Data from the annual American Time Use Survey, available from 2003, show that the largest increase in any category has been recorded in the "relaxing and leisure" category. A breakdown of the increase reveals that this rise is more-than-accounted-for by changes in the categories most directly related to leisure technologies, such as watching TV or using a computer (Figure A.4).

There are reasons why the time use survey data may underestimate the time that actually spent on modern leisure technologies, and perhaps overestimate the time spent working (or at least working attentively). First, the survey aims to uncover a person's

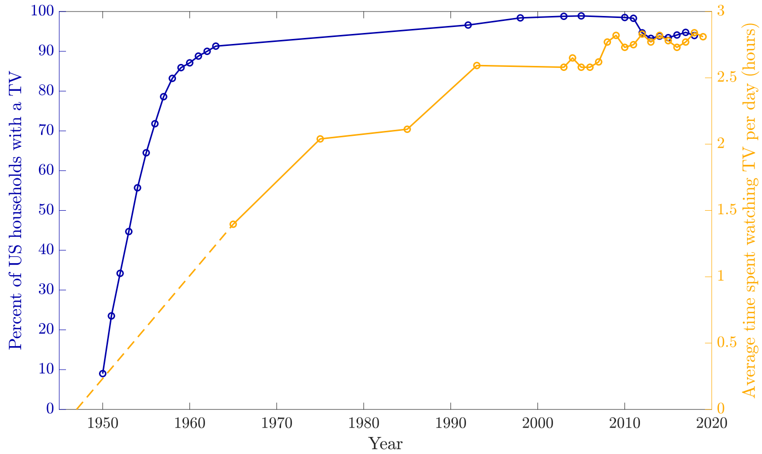

Figure A.3

Adoption of TV in the United States

Sources: American Time Use Survey, Aguiar and Hurst (2007b) and Comin and Hobijn (2009). Notes: the dashed line joins the first point available in the data on time use (1965) together with 1947, when fewer than 0.5% of households had a TV set installed at home – a proportion clearly too limited to show up in average time use across the population (source: Televisor Monthly, 1948, accessed via http://earlytelevision.org/us_tv_sets.html).

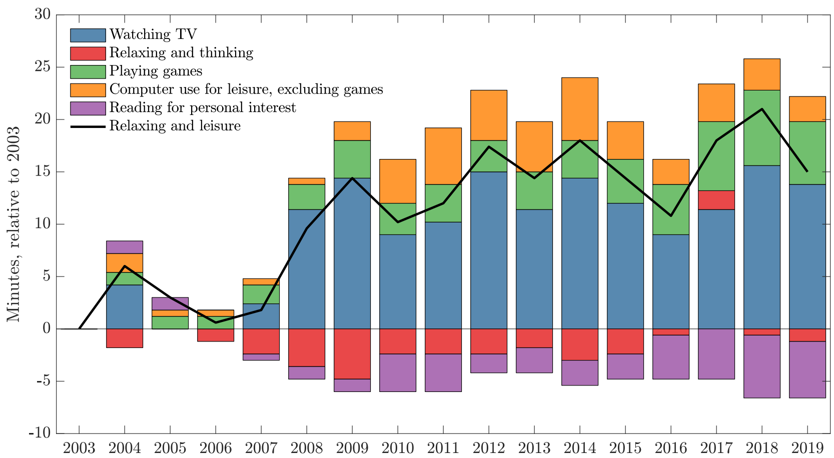

Figure A.4

Decomposition of the Increase in Relaxing and Leisure – the Category in the American Time Use Survey that has Experienced the Largest Increase Since 2003

Source: American Time Use Survey.

main activity at any given point in time during a day, and so if some of the leisure technologies are used during other activities (for example during work hours), their use will not be recorded. This is important since the evidence (which I discuss below) suggests that smartphones in particular are being used with a constant frequency throughout the day, including during work hours, and some of that use is likely to be related to leisure. For the same reason the BLS acknowledges that ATUS is not a good source of information on time spent online and/or using a computer or a smartphone: the survey is designed in such a way that time is split across many traditional categories such as working, socializing, etc.[57] This could give a misleading steer on the use of the leisure technologies if, for example, socializing today is different to socializing in the past (in particular if socializing today involved the use of leisure technologies). A related point is that, since people tend to check their phone very frequently (numerous estimates available online suggest that we pick-up our phones between 50 and 100 times a day), it is likely that the responders under-report usage when they fill in the survey. Consistent with that, some anecdotal evidence and the popularity of screen-time-tracking software suggests that users may find it hard to control the frequency of use and overall amount of time they spend on their devices. That could suggest possible underreporting in the surveys.

Given these possible shortcomings of the time-use survey data, the device-tracking data from Nielsen offers useful cross-check (even as it is not without drawbacks). The data paints a picture of a much more dramatic changes in time-use linked to technology (Figure A.5). For example, the data suggest that the amount of time spent on a smartphone *more than quadrupled* over the last 7 years, reaching over 3 hours daily. One of the limitations of these data is that they are not additive: a person can engage in multiple activities at once (e.g. watching TV and engaging on social media on the smartphone). Another is that the time spent on the devices could be productive time. Nonetheless, these data are a useful complement to the traditional time use surveys which naturally struggle to capture the short-but-frequent spells of usage.

The evidence on how people spend time at work (and indeed how much work is being carried out at home) is imperfect. For that reason it is useful to consider experimental tracking data on the frequency of use of technology throughout the day. In one such study, Christensen *et al.* (2016) measured smartphone screen time over the course of an average day among a sample of 653 people in 2014 in the United States (Figure A.6). Time spent on the phone averaged 1 hour and 29 minutes per day, a little higher than what the Nielsen data suggest (which makes sense since the study included only users, while Nielsen aim to weight their results to capture non-adopters). Most strikingly, the mobile phone usage appears to be uniformly distributed throughout the day, suggesting

[57] See https://www.bls.gov/tus/atusfaqs.htm#24 for the discussion of this point.

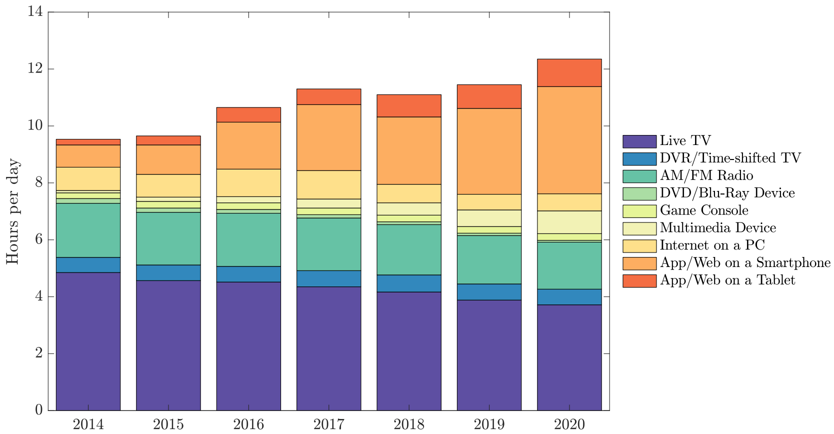

Figure A.5
Average Time Spent on Media Consumption per Adult in the US

Source: Nielsen. Note: Figures for representative samples of total US population (whether or not have the technology). More than one technology may be used at any given time, thus the total is indicative only. Data on TV and internet usage, and the usage of TV-connected devices are based on 248,095 individuals in 2016 and similar sample sizes in other years. Data on radio are based on a sample of around 400,000 individuals. There are approximately 9,000 smartphone and 1,300 tablet panelists in the U.S. across both iOS and Android smartphone devices.

that leisure time is, in part, substituting for time spent working. In a different study, Wallsten (2013) uses time use surveys to estimate that each minute spent on the internet is associated with loss of work-time of about 20 seconds.

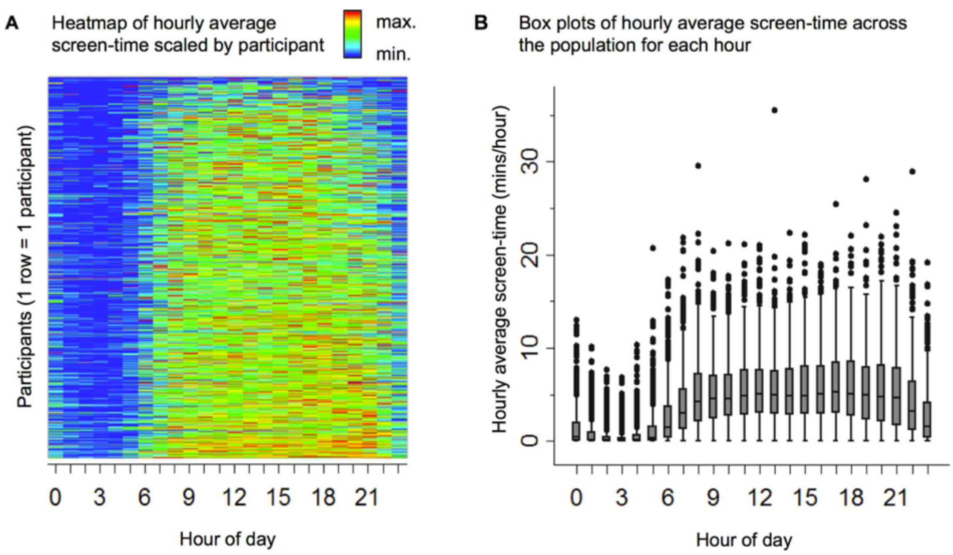

Figure A.6
Mobile Phone Use Over the Course of an Average Day

Source: Christensen *et al.* (2016).

Indeed, one feature of the latest technology is that it allowed leisure to "compete" with work much more directly than has been the case in the past. While it may not

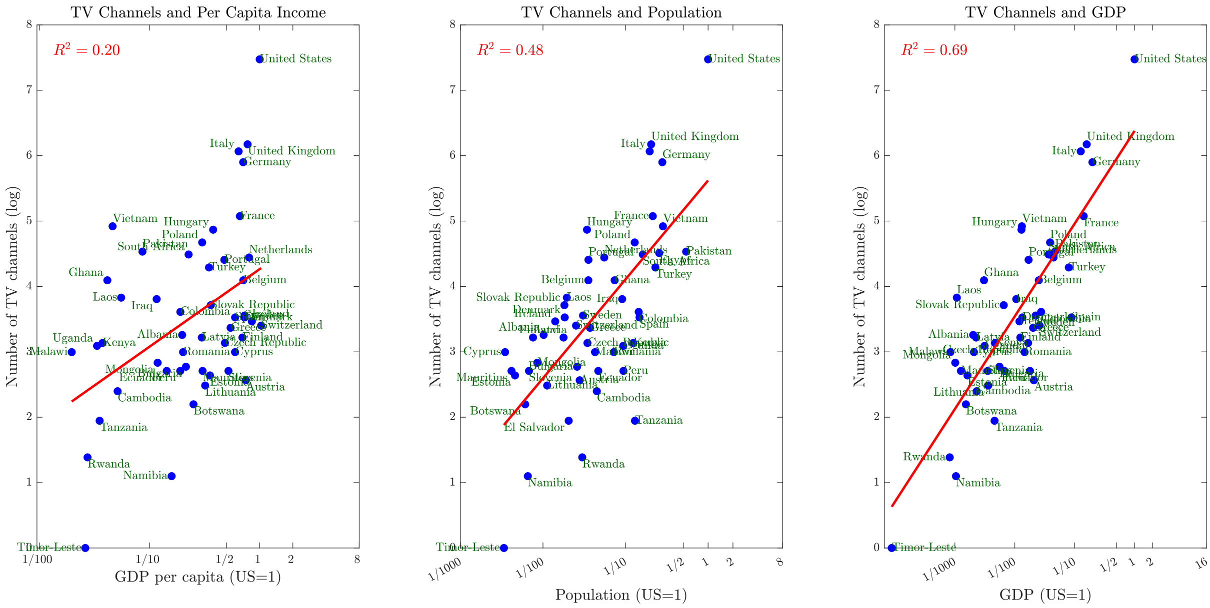

Figure A.7

The Number of TV Channels and Market Size

Sources: Data on GDP per capita are from Penn World Tables 9.0 (Feenstra *et al.* (2015)). Data on population are from the World Bank. Data on the number of TV channels in each country has been hand collected from online sources. As such these data are subject to some measurement error. TV channels include state-run channels.

have been possible to watch TV at work, online entertainment is available during the work hours. This is, at least in part, balanced by the possibility of accessing work emails at home. Future research should consider ways of measuring in more detail how people spend time at work and how much work is done at home.

Cross-country evidence on the market-size effect

In light of the theory, the equilibrium level of leisure technologies depends on market size. Figure A.7 provides a simple test of this prediction, by plotting the number of TV channels across countries against GDP per capita, population and the level of aggregate GDP separately in the three consecutive panels. The number of TV channels is potentially a useful metric of M in the context of cross-country analysis, because of the language- and culture- barriers tend to limit the market to national borders.[58] The rising R^2 from the left to the right panel suggests that market size, both in terms of level of development and population – does indeed play an important role.

[58]For some other leisure technologies such as mobile phone apps the market is global and cross-country exercise may be less useful. This concern could also apply to the English-speaking countries in the case of TV.

B Proofs and derivations for Sections 2, 4 and 5.

B.1 Solving the model of Section 2, incl. the proof of Proposition 1

I drop the (t) notation wherever this does not cause confusion.

B.1.1 Production

Decision problems on the production side of the economy are standard. Labor demand from the final goods producers is

$$w = (1-\alpha)\frac{Y}{L_Y}. \tag{71}$$

Solutions to final goods producers' and intermediate firms' problems give prices, quantities and profits that are identical across all intermediate firms:

$$p = \frac{r+\delta}{\alpha} \qquad x = \left(\frac{\alpha^2}{r+\delta}\right)^{\frac{1}{1-\alpha}} L_Y \qquad \pi = \alpha\frac{Y}{A}(1-\alpha). \tag{72}$$

B.1.2 R&D

R&D producers extract all the surplus from the intermediate firms. The price of a blueprint is given by

$$V(t) = \int_t^\infty e^{-\int_t^s r(u)du}\pi(s)ds. \tag{73}$$

Differentiating (73) with respect to time we obtain the no-arbitrage (Bellman) equation

$$\dot{V} = Vr - \alpha\frac{Y}{A}(1-\alpha). \tag{74}$$

Wages are equalized across sectors, and free entry to R&D implies

$$wL_A = \dot{A}V. \tag{75}$$

B.1.3 Household problem

Using (9), the representative consumer's problem can be written as

$$\max_{\{c,h\}_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t} \left(\log c + M(1-h)\right) dt \quad \text{subject to} \tag{76}$$

$$\dot{K}(t) = whN + rK + A\pi - V\dot{A} - cN. \tag{77}$$

The economy admits a representative household. The Hamiltonian associated with the household problem is:

$$\mathcal{H}(K, C, h; \mu) = \log c + M^{\zeta}(1-h) + \mu \left(whN + rK + A\pi - V\dot{A} - cN \right).$$

where $A\pi$ are the profits of the intermediate sector and $V\dot{A}$ are the flow purchases of the patents from the R&D sector.

The necessary conditions for an interior optimum are:

$$u_c = \mu N \tag{78}$$

$$u_h = -\mu w N \tag{79}$$

$$\rho\mu - \dot{\mu} = \mu r. \tag{80}$$

The transversality condition is:

$$\lim_{t\to\infty} e^{-\rho t} \mu(t) \cdot \left(K(t) + \int_0^{A(t)} V(i,t) di \right) = 0. \tag{81}$$

Equation (80) implies $\frac{\dot{\mu}}{\mu} = -(r - \rho)$. Integrating this equation we obtain $\mu(t) = u'(c_0) \exp\left(-\int_0^t (r(s) - \rho)\, ds \right)$. Substituting in (81) we can write the the transversality condition as

$$\lim_{t\to\infty} \left[\exp\left(-\int_0^t r(s) ds \right) \cdot \left(K(t) + \int_0^{A(t)} V(i,t) di \right) \right] = 0. \tag{82}$$

The intertemporal choice is standard. Differentiating equation (78) with respect to time and using (80) gives the individual Euler equation: $\frac{\dot{c}}{c} = r - \rho - n$. Since $\frac{\dot{C}}{C} = \frac{\dot{c}}{c} + n$, the Euler equation in terms of aggregate consumption is

$$\frac{\dot{C}}{C} = r - \rho. \tag{83}$$

Combining (78) and (79) gives $u_c w = -u_h$ and so $\frac{w}{c} = M^{\zeta}$. Multiplying and dividing by h, defining $\Phi := \frac{wh}{c}$ and noting hours worked are bound from above by the time endowment of 1 yields equation (10) in the main text.

B.1.4 Equilibrium

Definition 4. A *decentralized equilibrium in the economy with exogenous M* is a set of quantities and prices such that equations (6), (7) and (71) – (83) hold. A balanced growth equilibrium is an equilibrium where all model quantities grow at constant rates.

B.1.5 Growth along the balanced growth path – proof of Proposition 1.

Equation (77) implies that on a BGP labor income and consumption grow at the same rate, and thus $\Phi(t)$ is a constant and hours grow at the rate specified in equation (11). Differentiating the R&D production function (6) with respect to time, we obtain (12) and (13).

Along the BGP, consumption per capita grows at rate of labor income, and wages grow at the same rate as traditional productivity. Thus: $\gamma_c = \gamma_A + \gamma_h = \frac{n-\zeta(2-\phi)\gamma_M}{1-\phi}$. Leisure utility $\ell = M^\zeta(1-h)$ grows at rate $\zeta\gamma_M$ asymptotically. Thus utility is finite as long as $\rho - \zeta\gamma_M > 0$, as indicated in footnote 18.

The transversality condition in steady state is

$$\lim_{t\to\infty}\left[e^{-rt}\cdot(K(t)+A(t)V(t))\right]=0$$

Since $V(t)$ grows at a rate $\gamma_Y - \gamma_A$ on a BGP (see equation (74)), the transversality condition is satisfied as long as $r > \gamma_Y$. This is satisfied as long as $\rho > 0$, since (83) implies that in steady state $r = \rho + \gamma_Y$. This concludes the proof of Proposition 1.

B.2 Proof of Proposition 4

We can drop constraint (24) when solving this problem. Focusing on the interior solutions, the two optimality conditions are:

$$u_c \cdot \frac{1}{N}\frac{\partial Y}{\partial h} + u_h + u_M\frac{\partial M}{\partial h} = 0 \tag{84}$$

$$u_c \cdot \frac{1}{N}\frac{\partial Y}{\partial s_M} + u_M\frac{\partial M}{\partial s_M} = 0. \tag{85}$$

Since $u(c,h,M) = \frac{\left(c^{1-\epsilon}M^{\epsilon}v\left(hM^{\zeta}\right)\right)^{1-\sigma}}{1-\sigma}$, letting $\Theta := c^{1-\epsilon}M^{\epsilon}v\left(hM^{\zeta}\right)$ we have:

$$u_c = \Theta^{-\sigma}c^{-\epsilon}M^{\epsilon}(1-\epsilon)v\left(x\right) \tag{86}$$

$$u_h = \Theta^{-\sigma}c^{1-\epsilon}M^{\epsilon}v'\left(x\right)\frac{x}{h} \tag{87}$$

$$u_M = \Theta^{-\sigma}c^{1-\epsilon}M^{\epsilon-1}\left(\epsilon v\left(x\right) + \zeta v'(x)x\right) \tag{88}$$

Therefore:

$$\frac{u_M}{u_c}N = N\frac{c}{M}\frac{\epsilon v\left(x\right)+\zeta v'(x)x}{(1-\epsilon)v\left(x\right)} = \frac{C}{M}\frac{\epsilon+\zeta\varepsilon_v}{1-\epsilon} \tag{89}$$

$$\frac{u_h h}{u_c} = c\frac{v'\left(x\right)x}{(1-\epsilon)v\left(x\right)} = c\frac{\varepsilon_v}{1-\epsilon} \tag{90}$$

$$-\frac{u_h h}{u_M} = -M\frac{v'\left(x\right)x}{\epsilon v\left(x\right)+\zeta v'(x)x} = -M\frac{\varepsilon_v}{\epsilon+\zeta\varepsilon_v} \tag{91}$$

where $\varepsilon_v := \frac{v'(x)}{v}x$ is the elasticity of function v.

Since $L_Y = hN(1-s_M)$, $\frac{\partial h}{\partial L_Y} = \frac{1}{N(1-s_M)}$ and $\frac{\partial L_Y}{\partial s_M} = -hN$, we have that:

$$\frac{\partial Y}{\partial h} = -\frac{\partial Y}{\partial s_M}\frac{1-s_M}{h}$$

$$\frac{\partial M}{\partial h} = \frac{\partial M}{\partial s_M}\frac{s_M}{h}$$

Using these in the optimality conditions (84) and (85), we obtain:

$$-u_c\cdot\frac{1}{N}\frac{\partial Y}{\partial s_M}\frac{1-s_M}{h} + u_h + u_M\frac{\partial M}{\partial s_M}\frac{s_M}{h} = 0$$

$$u_c\cdot\frac{1}{N}\frac{\partial Y}{\partial s_M} + u_M\frac{\partial M}{\partial s_M} = 0$$

Together with (85) these imply:

$$-u_h h = u_M\frac{\partial M}{\partial s_M}$$

$$u_h h = u_c\frac{1}{N}\frac{\partial Y}{\partial s_M}$$

Using (90) and (91), these two equations become:

$$-\frac{\varepsilon_v}{\epsilon+\zeta\varepsilon_v} = \frac{1}{s_M} \tag{92}$$

$$-\frac{\varepsilon_v}{1-\epsilon} = \frac{1-\alpha}{1-s_M} \tag{93}$$

This is a system of two equations in two unknowns. Eliminating s_M and solving out for the elasticity we obtain:

$$\varepsilon_v = -\frac{1-\alpha(1-\epsilon)}{1+\zeta}. \tag{94}$$

Using this in (93) we obtain the expression (27) given in the proposition.

Expression (94) implies that when the optimal allocation is interior in both s_M and h, hM^ζ is independent of N: the elasticity is constant irrespective of what N is in the region where the optimum is interior. In such an optimal allocation, we have $hM^\zeta = h(s_M hN)^\zeta = h^{1+\zeta}\left(s_M^{SP}\right)^\zeta N^\zeta$. This is independent of N if $h^{1+\zeta} = \upsilon N^{-\zeta}$ for some constant υ. We have that hours are interior, $h \leq 1$, if $\upsilon N^{-\frac{\zeta}{1+\zeta}} \leq 1$ or equivalently when $N \geq \upsilon^{\frac{1+\zeta}{\zeta}} =: \bar{N}$. When the optimum is interior we can write equation (94) as $\varepsilon_v\left(\upsilon \cdot \left(s_M^{SP}\right)^\zeta\right) = -\frac{1-\alpha(1-\epsilon)}{1+\zeta}$. If an inverse of ε_v exists, this equation can be solved for υ:

$$\upsilon = \varepsilon_v^{-1}\left(-\frac{1-\alpha(1-\epsilon)}{1+\zeta}\right) \cdot \left(\frac{\frac{\epsilon}{1-\epsilon} - (1-\alpha)\zeta}{\frac{\epsilon}{1-\epsilon} + (1-\alpha)}\right)^{-\zeta} = \varepsilon_v^{-1}\left(-\frac{1-\alpha(1-\epsilon)}{1+\zeta}\right) \cdot \left(s_M^{SP}\right)^{-\zeta}.$$

For h to be interior, N must satisfy

$$N \geq \bar{N} = \left(\varepsilon_v^{-1}\left(-\frac{1-\alpha(1-\epsilon)}{1+\zeta}\right) \cdot \left(\frac{\frac{\epsilon}{1-\epsilon} - (1-\alpha)\zeta}{\frac{\epsilon}{1-\epsilon} + (1-\alpha)}\right)^{-\zeta}\right)^{\frac{1+\zeta}{\zeta}}.$$

Hours worked are

$$h = \left[\varepsilon_v^{-1}\left(-\frac{1-\alpha(1-\epsilon)}{1+\zeta}\right)\right]^{\frac{1}{1+\zeta}} \cdot \left(s_M^{SP} N\right)^{-\frac{\zeta}{1+\zeta}}.$$

Letting $\Delta := \left(\varepsilon_v^{-1}\left(-\frac{1-\alpha(1-\epsilon)}{1+\zeta}\right)\right)^{\frac{1}{\zeta}}$ we obtain (28) in the proposition.

The optimal level of s_M when $h = 1$. When $N < \bar{N}$, h^{SP} will be at the corner and equal to 1. I now derive the optimal s_M in this case. Note that while (84) no longer holds with equality, (85) continues to hold. Therefore, using (89) we have:

$$\frac{s_M}{1-s_M} = \frac{1}{1-\alpha}\frac{\epsilon + \zeta\varepsilon_v\left((s_M N)^\zeta\right)}{1-\epsilon},$$

Since $h = 1$, for a given N ε_v is a function of s_M only. If a solution to this equation exists, it represents the optimal s_M^{SP}. In this case, then, s_M^{SP} is in general a function of N.

B.3 Proof of Proposition 5

Suppose there is a positive measure of active platforms, so that $\Omega = 1$. This is only the case in equilibrium if the choice of hours is interior, $h < 1$. The free entry condition (43) combined with demand for brand equity (32) give $\alpha^2\chi Y = wL_M$. Substituting in for wages using the demand for labor from the final goods producers, plugging in the expressions from the definitions of L_Y and L_M and solving for s_M yields expression (44) in the proposition.

Next, consider labor supply (still assuming interior solution for hours worked). Substituting the budget constraint into (38) and re-arranging, we obtain:

$$h = -\frac{u_c\left(c - rk - \tilde{\pi}\right)}{u_h}. \tag{95}$$

Each intermediate producer sets identical prices, produces the same quantity and purchases the same quantity of brand equity, resulting in identical profits:

$$p = \frac{r}{\alpha} \qquad x = \left(\frac{\alpha^2}{r}\right)^{\frac{1}{1-\alpha}} L_Y \qquad \pi = \alpha\frac{Y}{A}(1-\alpha-\alpha\chi).$$

Thus each household receives profits of $\tilde{\pi} = \alpha\frac{Y}{N}\left(1-\alpha-\alpha\chi\right)$. Each household also earns capital income $rk = \alpha^2\frac{Y}{N}$. Let $y := \frac{Y}{N}$ denote per capita output. Total non-labor income per-capita is $rk + \tilde{\pi} = \alpha y\left(1-\alpha\chi\right)$. Market clearing implies $y = c$ and so equation (95) becomes

$$h = -\frac{u_c c\left(1-\alpha+\alpha^2\chi\right)}{u_h}. \tag{96}$$

By (90) we have:

$$\frac{u_c c}{u_h} = (1-\epsilon)\frac{h}{\varepsilon_v}. \tag{97}$$

Using (97) in (96) gives equation (**??**) in the proposition: $\varepsilon_v = -(1-\epsilon)(1-\alpha+\alpha^2\chi)$. We have $h^{1+\zeta}(s_M N)^{\zeta} = \varepsilon_v^{-1}\left(-(1-\epsilon)(1-\alpha+\alpha^2\chi)\right)$. Defining $\tilde{\Delta} := \left(\varepsilon_v^{-1}\left(-(1-\epsilon)(1-\alpha+\alpha^2\chi)\right)\right)^{\frac{1}{\zeta}}$ and solving for h we obtain

$$h = \left(\frac{\tilde{\Delta}}{s_M^{DC} N}\right)^{\frac{\zeta}{1+\zeta}}.$$

Finally, consider when $h < 1$ and $s_M^{DC} > 0$. Clearly, h is interior when $\left(\frac{\left(1+\frac{1-\alpha}{\alpha^2\chi}\right)\tilde{\Delta}}{N}\right)^{\frac{\zeta}{1+\zeta}} \leq 1$, or, equivalently, when $N \geq \tilde{N} := \left(1+\frac{1-\alpha}{\alpha^2\chi}\right)\tilde{\Delta}$. Conversely, when $N < \tilde{N}$, then $h^{DC} = 1$ and no platforms enter: $\Omega = B = s_M^{DC} = 0$.

B.4 Solving the dynamic model of Section 5, and proofs of Propositions 8, 9 and 10

B.4.1 Household problem

The Hamiltonian and the necessary conditions are the same as in Section B.1.3 above. The only difference is that now u_c might be a function of h and M as well. Differentiating equation (86) we obtain the expressions for the cross-derivatives:

$$u_{cc} = -\frac{u_c}{c}\left(\epsilon + (1-\epsilon)\sigma\right) \tag{98}$$

$$u_{ch} = \frac{u_c}{h}(1-\sigma)\varepsilon_v \tag{99}$$

$$u_{cM} = \frac{u_c}{M}\left(\epsilon\left(1-\sigma\right) + (1-\sigma)\zeta\varepsilon_v\right) \tag{100}$$

Differentiating equation (78) with respect to time gives

$$\frac{u_{cc}c}{u_c}\frac{\dot{c}}{c} + \frac{u_{ch}h}{u_c}\frac{\dot{h}}{h} + \frac{u_{cM}M}{u_c}\frac{\dot{M}}{M} = n + \frac{\dot{\mu}}{\mu}. \tag{101}$$

Plugging in the results from (98)-(100) and (86) into (101), using equation (80) and rearranging we obtain the individual consumption Euler Equation:

$$\frac{\dot{c}}{c} = \frac{1}{\epsilon + (1-\epsilon)\sigma}\left[(r-\rho-n) + (1-\sigma)\varepsilon_v\frac{\dot{h}}{h} + (1-\sigma)\left(\epsilon + \zeta\varepsilon_v\right)\frac{\dot{M}}{M}\right] \tag{102}$$

Since $\frac{\dot{C}}{C} = \frac{\dot{c}}{c} + n$, the Euler equation in terms of aggregate consumption is

$$\frac{\dot{C}}{C} = \frac{1}{\epsilon + (1-\epsilon)\sigma}\left[(r-\rho-n) + (1-\sigma)\varepsilon_v\frac{\dot{h}}{h} + ((1-\sigma)\left(\epsilon + \zeta\varepsilon_v\right))\frac{\dot{M}}{M}\right] + n \tag{103}$$

Note that with log consumption utility ($\sigma = 1$), the aggregate Euler equation simplifies to the familiar $\frac{\dot{C}}{C} = r - \rho$.

Combining (78) and (79) gives $-\frac{u_h}{u_c} = w$. Substituting in for wages using the labor demand from the final goods sector we obtain: $-\frac{u_h h}{u_c} = (1-\alpha)\frac{Y}{(1-s_A-\Omega s_M)N}$. Now substitute for the left hand side using equations (86) and (87) to obtain:

$$\varepsilon_v = -(1-\epsilon)\frac{1-\alpha}{1-s_A-\Omega s_M}\frac{Y}{C}. \tag{104}$$

B.4.2 Production and profits in the intermediate sector

The solution is the same as in Section B.3 except the cost of capital $r + \delta$ includes depreciation. Thus:

$$\bar{p} = \frac{r+\delta}{\alpha} \qquad \bar{x} = \left(\frac{\alpha^2}{r+\delta}\right)^{\frac{1}{1-\alpha}} L_Y \qquad \bar{\pi} = \alpha\frac{Y}{A}(1-\alpha-\alpha\chi). \tag{105}$$

B.4.3 Equilibrium in the brand equity market

In equilibrium $b_i = \bar{b}\forall i$ and hence the price of brand equity satisfies

$$p_B = \alpha^2\chi\frac{Y}{B}. \tag{106}$$

The current-value Hamiltonian that corresponds to platform j's problem is

$$\mathcal{H} = p_B \cdot M_j\frac{\ell}{M} - wL_{M,j} + Z\left[L_{M,j}A^{\phi}\right]$$

where Z is the costate variable, or the shadow value of leisure blueprint. By the Maximum Principle, the solution satisfies (58) and (59). Plugging (106) into (59) yields the Bellman equation for the shadow value of a leisure blueprint:

$$\dot{Z} = rZ - \alpha^2\chi\frac{Y}{M}. \tag{107}$$

B.4.4 Equilibrium prices of traditional blueprints

Given profits in (105), the Bellman equation for the value of traditional blueprint is

$$\dot{V} = Vr - \alpha\frac{Y}{A}(1-\alpha-\alpha\chi). \tag{108}$$

B.4.5 Free entry

Free entry into the traditional and leisure R&D implies

$$wL_A = V\dot{A}$$
$$wL_M = Z\dot{M}.$$

Substituting in for $\dot{A}$ and $\dot{M}$ from the R&D technologies gives:[59]

$$
\begin{aligned}
w &= VA^{\phi} \\
Z &= V.
\end{aligned}
$$

B.4.6 Equilibrium conditions

Collecting the equilibrium conditions, we have:

$$\dot{K} = Y - C - \delta K \tag{109}$$

$$\dot{A} = s_A h N A^{\phi} \tag{110}$$

$$\dot{M} = \Omega \cdot s_M h N A^{\phi} \tag{111}$$

$$\dot{V} = rV - \alpha \frac{Y}{A}\left(1 - \alpha - \Omega \cdot \alpha\chi\right) \tag{112}$$

$$\dot{Z} = \Omega \cdot \left(rZ - \alpha^2 \chi \frac{Y}{M}\right) \tag{113}$$

$$\frac{\dot{C}}{C} = \frac{(r - \rho - n) + (1 - \sigma)\left(\varepsilon_v \frac{\dot{h}}{h} + (\epsilon + \zeta\varepsilon_v)\frac{\dot{M}}{M}\right)}{\epsilon + (1 - \epsilon)\sigma} + n \tag{114}$$

$$\varepsilon_v = -(1 - \epsilon)\frac{1 - \alpha}{1 - s_A - \Omega s_M}\frac{Y}{C} \tag{115}$$

$$Y = K^{\alpha}(A(1 - s_A - \Omega s_M)hN)^{1-\alpha} \tag{116}$$

$$r = \alpha^2 \frac{Y}{K} - \delta \tag{117}$$

$$w = (1 - \alpha)\frac{Y}{(1 - s_A - \Omega s_M)\, hN} \tag{118}$$

$$w = VA^{\phi} \tag{119}$$

$$Z = V \tag{120}$$

$$B = 1 - h \tag{121}$$

B.4.7 Steady state growth rates

The resource constraint (109) and the aggregate production function (116) imply that s_A and s_M must be constant on the balanced growth path. Dividing equation (110) through

[59] With asymmetric spillovers, $\phi_A \neq \phi_M$, these two equations would read $w = VA^{\phi_A}$ and $Z = VA^{\phi_A - \phi_M}$. The second of these implies $\gamma_Z = \gamma_V + (\phi_A - \phi_M)\gamma_A$. To see how this is consistent with balanced growth, note that the Bellman equations for V and Z imply that the ratios $\frac{Y}{MZ}$ and $\frac{Y}{AV}$ are constant, so that on the BGP $\gamma_V + \gamma_A = \gamma_Z + \gamma_M$. Combining these results, we get $\gamma_M = \gamma_A(1 - \phi_A + \phi_M)$. This is also the relationship between the long-run growth rates of the two technologies that emerges from differentiating the two ideas production functions with respect to time.

by A and differentiating with respect to time we obtain $\gamma_A = \frac{\gamma_h + n}{1-\phi}$. Dividing equation (111) through by M and differentiating with respect to time we get $\gamma_M = \gamma_h + n + \phi\gamma_A$. Since hM^{ζ} is constant on the BGP, we also have $\gamma_h = -\Omega\zeta\gamma_A$. This yields a system of 3 linear equations in as many unknowns. The solution is $\gamma_A = \gamma_M = \frac{n}{1-\phi+\Omega\zeta}$ and $\gamma_h = -\Omega\zeta\frac{n}{1-\phi+\Omega\zeta}$, as in the proposition. The formula for the growth of per capita output and consumption follows from (109) and (116).

B.4.8 Steady state employment shares

Along segment 2 of the sBGP, equations (112), (113) and (120) imply

$$\frac{\alpha^2\chi\frac{Y}{M}}{r-(\gamma_Y-\gamma_A)} = \frac{\alpha\frac{Y}{A}(1-\alpha-\alpha\chi)}{r-(\gamma_Y-\gamma_A)}$$

which, together with (110) and (111) give

$$\frac{s_A}{s_M} = \frac{A}{M} = \frac{1-\alpha-\alpha\chi}{\alpha\chi}. \tag{122}$$

Next, equation (119) implies:

$$(1-\alpha)\frac{s_A}{1-s_A-s_M} = \alpha(1-\alpha-\alpha\chi)\frac{\gamma_A}{\rho+\gamma_A} \tag{123}$$

We have two equations – (122) and (123) – in two unknowns. Solving out for s_A and s_M yields the formulas in Proposition 9. Finally, along segment 1, equation (123) reads instead $(1-\alpha)\frac{s_A}{1-s_A} = \alpha(1-\alpha)\frac{\gamma_1}{\rho+\gamma_1}$ which readily gives the formula in the proposition.

B.4.9 Transition dynamics

Proposition 10. *Let $\gamma_A := n\frac{1}{1-\phi+\Omega\zeta}$, $\gamma_Y := \gamma_A(1-\Omega\zeta)+n$ denote the steady state growth rates of A and Y along the two segments of the sBGP. Define $\beta_A := \gamma_A/n$* and *$\beta_Y := \gamma_Y/n$. The stationary variables are then defined as follows: $k := \frac{K}{N^{\beta_Y}}$, $a := \frac{A}{N^{\beta_A}}$, $m := \frac{M}{N^{\Omega\beta_A}}$, $v := \frac{V}{N^{\beta_Y-\beta_A}}$, $z := \frac{Z}{N^{\Omega(\beta_Y-\beta_A)}}$ $\tilde{c} := \frac{C}{N^{\beta_Y}}$, $\tilde{h} := \frac{h}{N^{-\Omega\zeta\beta_A}}$, $y := \frac{Y}{N^{\beta_Y}}$, $\tilde{w} = \frac{w}{N^{\beta_A}}$. The* *dynamic equilibrium is the solution to the system:*

$$\dot{k} = y - \tilde{c} - \delta k - \gamma_Y k \tag{124}$$

$$\dot{a} = a^{\phi} s_A \tilde{h} - \gamma_A a \tag{125}$$

$$\dot{m} = \Omega \left(a^{\phi} s_M \tilde{h} - \gamma_A m \right) \tag{126}$$

$$\dot{v} = v \left(r - (\gamma_Y - \gamma_A) \right) - \alpha(1 - \alpha - \Omega\alpha\chi) \frac{y}{a} \tag{127}$$

$$\dot{z} = \Omega \left(z \left(r - (\gamma_Y - \gamma_A) \right) - \alpha^2 \chi \frac{y}{m} \right) \tag{128}$$

$$\frac{\dot{\tilde{c}}}{\tilde{c}} = \frac{(r - \rho - n) + (1 - \sigma) \left[\varepsilon_v \cdot \left(\frac{\dot{\tilde{h}}}{\tilde{h}} - \zeta \gamma_A \right) + (\epsilon + \zeta \varepsilon_v) \left(\frac{\dot{m}}{m} + \gamma_A \right) \right]}{\epsilon + (1 - \epsilon)\sigma} + n - \gamma_Y \tag{129}$$

$$\frac{\tilde{c}}{y} = -(1 - \epsilon) \frac{1 - \alpha}{1 - s_A - \Omega s_M} \frac{1}{\varepsilon_v(\tilde{h} m^{\zeta})} \tag{130}$$

$$y = k^{\alpha} \left((1 - s_A - \Omega s_M) \tilde{h} a \right)^{1-\alpha} \tag{131}$$

$$r = \alpha^2 \frac{y}{k} - \delta \tag{132}$$

$$\tilde{w} = (1 - \alpha) \frac{y}{(1 - s_A - \Omega s_M)\, \tilde{h}} \tag{133}$$

$$\tilde{w} = v \cdot a^{\phi} \tag{134}$$

$$z = v. \tag{135}$$

The segmentation of the balanced growth path means that, even along the sBGP, the aggregate variables grow at different constant rates at different points in time. The proposition shows how we can still derive a stationary system that pins down the dynamic equilibrium. The indicator variable $\Omega \in \{0, 1\}$ acts as a switch between the two segments of the sBGP (recall that $\Omega = 0$ if no platforms are active in equilibrium). It shows up in the definitions of the stationary variables, capturing the fact that the trend growth rates change between the two segments. In particular, variables such as M, Z and h are constant in segment 1, so that they are equal to their de-trended counterparts when $\Omega = 0$. Ω also shows up in the dynamic equations: $\Omega = 0$ effectively removes the leisure-R&D from the system. In other words, with $\Omega = 0$ the system collapses to the textbook model of semi-endogenous growth; with $\Omega = 1$ it represents the full dynamic equilibrium with active platforms and leisure-R&D.[60]

Proof. First note that r, s_A and s_M are asymptotically constant. Taking logs and differentiating the expression which defines k with respect to time gives $\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \gamma_Y$, or

[60]Note that the equilibrium system (124)-(135) does not explicitly feature brand equity or its price. Given the solution to the system above, these variables are uniquely pinned down by the brand equity production function and the aggregate demand for brand equity: $B = N(1 - h)$ and $p_B = \alpha^2 \chi \frac{Y}{B}$.

$\dot{K} = \frac{\dot{k}}{k}K + \gamma_Y K = \dot{k}N^{\beta_Y} + \gamma_Y K$. Therefore (109) can be written as:

$$\dot{k}N^{\beta_Y} + \gamma_Y K = yN^{\beta_Y} - \tilde{c}N^{\beta_Y} - \delta k N^{\beta_Y}.$$

Dividing through by N^{β_Y} and rearranging yields equation (124). Similarly, since A grows at γ_A in the steady state, we have $\dot{A} = \dot{a}N^{\beta_A} + \gamma_A A$. Solving for $\dot{a}$ we get:

$$\dot{a} = \frac{A^\phi L_A}{N^{\beta_A}} - \gamma_A a = a^\phi N^{\beta_A \phi} s_A \tilde{h} N^{1-\zeta\beta_A} N^{-\beta_A} - \gamma_A a.$$

Proposition 8 implies that $\beta_A\phi - \zeta\beta_A + 1 - \beta_A = 0$. We thus obtain equation (125). In the same fashion, (111) gives (126). It follows from the definition of v that $\dot{V} = \dot{v}(N^{\beta_Y - \beta_A}) + (\gamma_Y - \gamma_A)V$. Plugging this into equation (112) yields (127). Following the same steps, from (113) we get $\dot{z}(N^{\beta_Y-\beta_A}) + (\gamma_Y - \gamma_A)vN^{\beta_Y-\beta_A} = rzN^{\beta_Y-\beta_A} - \alpha^2\chi\frac{y}{m}N^{\beta_Y-\beta_A}$ which yields (128). Next, we have $\dot{C} = \dot{\tilde{c}}N^{\beta_Y} + \gamma_Y C$. Using this in (114) gives (129). Using the definitions of stationary variables in (115), (116), (117) and (118) it is straightforward to derive (130), (131), (132) and (133), respectively. Plugging in the definitions of stationary variables into (119) we get: $\tilde{w}N^{\beta_A}\tilde{h}N^{1-\zeta\beta_A} = vN^{\beta_Y-\beta_A}\left(\dot{a}N^{\beta_A} + \gamma_A a N^{\beta_A}\right)$. Cancelling terms gives (134). Finally, using the definitions of stationary variables in equation (120) gives (135). □

B.5 Solution to the planner's problem

Proposition 11. *At any point in time, the optimal allocation is the solution to the system of equations:*

$$\dot{k} = y - \tilde{c} - \delta k - \gamma_Y k \tag{136}$$

$$\dot{a} = a^\phi s_A \tilde{h} - \gamma_A a \tag{137}$$

$$\dot{m} = a^\phi s_M \tilde{h} - \gamma_A m \tag{138}$$

$$\dot{v} = v\left(\alpha\frac{y}{k} - \delta - \phi\left(\frac{\dot{a}}{a} + \gamma_A\right) - (\gamma_Y - \gamma_A)\right) + z\phi\left(\frac{\dot{m}}{a} + \gamma_A\frac{m}{a}\right) - (1-\alpha)\frac{y}{a} \tag{139}$$

$$\dot{z} = z\left(\alpha\frac{y}{k} - \delta - (\gamma_Y - \gamma_A)\right) - \frac{c}{m}\frac{\epsilon + \zeta\varepsilon_v}{1-\epsilon} \tag{140}$$

$$\frac{\dot{\tilde{c}}}{\tilde{c}} = \frac{\left(\alpha\frac{y}{k} - \delta - \rho - n\right) + (1-\sigma)\left[\varepsilon_v \cdot \left(\frac{\dot{\tilde{h}}}{\tilde{h}} - \zeta\gamma_A\right) + (\epsilon + \zeta\varepsilon_v)\left(\frac{\dot{m}}{m} + \gamma_A\right)\right]}{\epsilon + (1-\epsilon)\sigma} + n - \gamma_Y \tag{141}$$

$$\varepsilon_v = -(1-\epsilon)\frac{1-\alpha}{1-s_A-s_M}\frac{y}{\tilde{c}} \tag{142}$$

$$y = k^\alpha\left((1-s_A-s_M)\tilde{h}a\right)^{1-\alpha} \tag{143}$$

$$v = (1-\alpha)\frac{y}{(1-s_A-s_M)\tilde{h}}a^{-\phi} \tag{144}$$

$$z = v \tag{145}$$

where all the variables are appropriately normalized so that they are constant on a BGP.

Proof. The Hamiltonian corresponding to the planning problem is

$$\mathcal{H} = u(c,h,M) + \mu_K\left(K^\alpha\left(A(1-s_A-s_M)hN\right)^{1-\alpha} - C - \delta K\right) + \mu_A\left(s_A h N A^\phi\right) + \mu_M\left(s_M h N A^\phi\right)$$

Interior optimum satisfies the usual optimality conditions:

$$u_c = N\mu_K \tag{146}$$

$$\mu_K(1-\alpha)\frac{Y}{1-s_M-s_A} = \mu_A\frac{\dot{A}}{s_A} = \mu_A hNA^{\phi} \tag{147}$$

$$\mu_K(1-\alpha)\frac{Y}{1-s_M-s_A} = \mu_M\frac{\dot{M}}{s_M} = \mu_M hNA^{\phi} \tag{148}$$

$$-u_h = \mu_K(1-\alpha)\frac{Y}{h} + \mu_A\frac{\dot{A}}{h} + \mu_M\frac{\dot{M}}{h} \tag{149}$$

$$\rho - \frac{\dot{\mu}_A}{\mu_A} = \phi\frac{\dot{A}}{A} + \frac{\mu_K}{\mu_A}(1-\alpha)\frac{Y}{A} + \frac{\mu_M}{\mu_A}\phi\frac{\dot{M}}{A} \tag{150}$$

$$\rho - \frac{\dot{\mu}_K}{\mu_K} = \alpha\frac{Y}{K} - \delta \tag{151}$$

$$\rho - \frac{\dot{\mu}_M}{\mu_M} = \frac{1}{\mu_M}u_M \tag{152}$$

Since $Y = K^{1-\alpha}(Ah(1-s_A-s_N)N)^{\alpha}$, on a BGP $\gamma_Y = (1-\alpha)\gamma_Y + \alpha(\gamma_A+\gamma_h+n)$ and so output grows at $\gamma_Y = \gamma_A+\gamma_h+n$. The resource constraint implies $\gamma_C = \gamma_c+n = \gamma_A+\gamma_h+n$ so that the growth of per-capita consumption in steady state is $\gamma_c = \gamma_A + \gamma_h$. Equation (149) and (90) together imply that $\gamma_h = -\zeta\gamma_M$. Using this in the ideas production function yields$\gamma_A = \gamma_M = \frac{n}{1-\phi+\zeta}$.

Combining (150) and (151) we obtain: $\frac{\dot{\mu}_A}{\mu_A} - \frac{\dot{\mu}_K}{\mu_K} = -\frac{\mu_K}{\mu_A}(1-\alpha)\frac{Y}{A} - \frac{\mu_M}{\mu_A}\phi\frac{\dot{M}}{A} + \alpha\frac{Y}{K} - \delta - \phi\frac{\dot{A}}{A}$. Divide through by $\frac{\mu_K}{\mu_A}$ and note that $\frac{\dot{\mu}_A}{\mu_K} - \frac{\mu_A}{\mu_K}\frac{\dot{\mu}_K}{\mu_K} = \frac{\dot{\mu}_A\mu_K - \mu_A\dot{\mu}_K}{\mu_K^2} = \dot{\left(\frac{\mu_A}{\mu_K}\right)}$. In parallel with the equilibrium notation, let $V := \frac{\mu_A}{\mu_K}$ and $Z := \frac{\mu_M}{\mu_K}$. Then:

$$\dot{V} = V\left(\alpha\frac{Y}{K} - \delta - \phi\frac{\dot{A}}{A}\right) - Z\left(\phi\frac{\dot{M}}{A}\right) - (1-\alpha)\frac{Y}{A}.$$

Note that V grows at $\gamma_Y - \gamma_A = n + \gamma_h$ on the socially optimal BGP. Similarly, combining (152) and (151) we obtain: $\dot{Z} = Z\left(\alpha\frac{Y}{K} - \delta\right) - \frac{u_M}{u_c}N$. Plugging in for the derivatives we get

$$\dot{Z} = Z\left(\alpha\frac{Y}{K} - \delta\right) - \frac{C}{M}\frac{\epsilon + \zeta\varepsilon_v}{1-\epsilon}.$$

Equations (147) and (148) yield

$$(1-\alpha)\frac{Y}{(1-s_M-s_A)\,Nh} = V \cdot A^{\phi} \tag{153}$$

$$V = Z \tag{154}$$

Finally, equation (149) gives

$$-\frac{u_h h}{u_c} N = (1-\alpha)Y + V\dot{A} + Z\dot{M}$$

Substituting in for the derivatives on the left-hand side and using the R&D technology equations with (153) and (154) gives

$$\varepsilon_v = -(1-\epsilon)\frac{1-\alpha}{1-s_M-s_A}\frac{Y}{C}.$$

Normalizing variables by their steady state growth rates delivers the system in the proposition. □

C Subsidies that decentralize the optimal allocation

Subsidy to intermediate goods production

Market power in the intermediate goods sector means that firms in that sector produce too little output. Standard arguments lead to the optimal subsidy of $\tau_x = \frac{1-\alpha}{\alpha}$ which can correct this distortion. With such a subsidy in place, the return to capital is at the efficient level: $\hat{r} = \alpha \frac{Y}{K} - \delta$ (I denote the variables under the subsidy scheme with a hat).

All else equal, this subsidy raises the demand for brand equity, which is $\hat{p}_B(B) = \alpha\chi\frac{Y}{B}$ with a subsidy, instead of (32) in a laissez faire equilibrium. The subsidy also raises profits:

$$\hat{\pi} = \frac{Y}{A}\left(1 - \alpha - \alpha\chi\right). \tag{155}$$

Note that the planner does not need to offset the brand equity spending of the intermediate firms directly, since ad spending does not affect the equilibrium price or quantity that existing firms supply to the market (the intuition is that brand equity spending is not part of the marginal cost of production). Instead, brand equity spending affects profitability, which has implications for incentives to engage in R&D.

Subsidies to traditional R&D

Private returns to R&D do not take into account the knowledge spillovers from innovation, including to leisure R&D. Additionally they are depressed by the brand equity spending ex-post (the $-\alpha\chi$ term in (155)). Subsidies to traditional R&D may be used to correct these distortions. In an equilibrium with τ_x already in place, the price of a traditional blueprint is given by $V(t) = \int_t^\infty e^{-R(s)}\hat{\pi}(s)ds$ where $R(s) := \int_t^s \hat{r}(u)du$. If subsidies $\tau_A(s)$ and $\tau_\chi(s)$ are introduced, inventors receive $\hat{V}(t) = \int_t^\infty e^{-R(s)}\left(\hat{\pi}(s) + \tau_A(s) + \tau_\chi(s)\right)ds$. Comparison of the Bellman equations that pin down the evolution of the value of the blueprint in the two allocations shows that setting $\tau_A = \hat{V}\phi\frac{\dot{A}}{A} + \hat{Z}\phi\frac{\dot{M}}{A}$ and $\tau_\chi = \alpha\chi\frac{Y}{A}$ (where $\hat{Z}$ is the shadow value of the leisure blueprint with subsidies to leisure R&D, discussed next, in place) equalizes private and social returns. What these expressions say is that, in the world with leisure technologies, R&D subsidies must correct not only for the usual within-sector knowledge externalities, but also for cross-sector spillovers and for the reduced ex-post profitability of the intermediate producers.

Subsidies to leisure R&D

The level of leisure R&D is inefficient, since leisure R&D activities are guided by the profitability of the brand equity business and not by the social value of leisure technologies, as discussed in detail in Section 5. The planner can correct for the indirect monetization distortion by subsidizing platforms. Let τ_M be the proportional subsidy to the sales of brand equity. The socially optimal allocation can be implemented by setting

$$\tau_M = \frac{C}{Y} \frac{1}{\alpha\chi} \frac{\epsilon + \zeta\varepsilon_v}{1 - \epsilon} - 1. \tag{156}$$

The subsidy is constant along the balanced growth path, and depends negatively on $\alpha\chi$ (since a higher $\alpha\chi$ brings the equilibrium supply of leisure technologies closer to the optimum) and positively on the weight of leisure technologies in utility ϵ. Note that, in line with the discussion in Section 4, when ϵ is sufficiently low, there might be too much leisure technologies in the laissez-faire equilibrium, and the optimal policy would be to tax leisure R&D.

Online supplementary material

D Proof of Propositions 2 and 3

D.1 Defining balanced growth preferences

Consider an economy with traditional and leisure technology growth, with gross growth rates of g_A and g_M, respectively. Household preferences are ordered by

$$U = \int_0^\infty e^{-\rho t} u(c, h, M) dt,$$

where u is the instantaneous utility function, ρ is the discount rate, $c := C/N$ is per capita consumption, h is hours worked and M is an index of leisure technology. We first define the class of functions that delivers balanced growth in a decentralized equilibrium and in the optimal allocation.

A balanced growth path is a long-run equilibrium in which output and consumption grow at constant rates. On any candidate BGP, hours worked must either be constant or must be declining at a constant rate: a positive growth rate of hours can be ruled out as hours worked would breach the time feasibility constraint in finite time; a non-constant growth rate of hours can be ruled out as it would lead to a non-constant growth of output, contradicting the definition of a BGP. In general, with the two sources of technological improvements, growth of hours worked on a BGP can be a function of the growth rate of traditional technology (through income and substitution effects, as in Boppart and Krusell (2020)) and leisure technology (as this technology might shift the marginal utility of leisure time). Therefore, for some non-negative numbers ϱ and ζ, we can write the gross growth rate of hours worked along a BGP as:

$$g_h = g_A^{-\varrho} g_M^{-\zeta}, \tag{157}$$

with$\varrho = \zeta = 0$ corresponding to the standard case with constant hours worked.[61]

The representative household budget constraint and the resource constraint imply that the growth of per-capita consumption is[62]

$$g_c = g_w g_h = g_A^{1-\varrho} g_M^{-\zeta}$$

[61] Strictly speaking, a BGP could exist even if one of these effects acts to raise hours worked, as long as the overall effect is that hours are non-increasing. I rule this cases out by assuming that both ϱ and ζ are non-negative.

[62] I rule out cases with negative or zero consumption growth along the BGP. If the two technologies grow at identical rates this assumption boils down to the restriction that $1 - \varrho - \zeta > 0$.

where the second equality uses (157) and incorporates the fact that wages grow at the rate of traditional technology. Moreover, on any candidate BGP, the shares of resources employed across the different sectors of the economy must be constant.

To formally define balanced growth preferences in the environment with traditional and leisure technology, it is useful to consider the optimal allocation, and in particular the necessary conditions for an interior social optimum, which is more demanding than the equilibrium with zero price leisure goods.

D.1.1 Optimal allocation in the long-run

The planner maximizes the lifetime utility of a representative household subject to the technological and feasibility constraints:

$$\max_{c,h,s_Y,s_A,s_M} \int_0^\infty e^{-\rho t} u(c,h,M)dt \tag{158}$$

subject to $\dot{K} = Y - C - \delta K$, $Y = K^\alpha (AL_Y)^{1-\alpha}$, $\dot{M} = L_M \cdot g(\cdot)$, $\dot{A} = L_A \cdot f(\cdot)$, $h + \ell = 1$, $L_Y + L_A + L_M = hN$, $N = N_0 e^{nt}$, where the shares are defined as $s_A = \frac{L_A}{hN}$, etc. The focus here is on preferences, so to simplify I assume, without loss of generality, that final good technology is Cobb-Douglas and the ideas production functions are power functions of labor with exponent equal to 1. The Hamiltonian for this problem is:

$$\mathcal{H} = u(c,h,M) + \mu_K (Y - C - \delta K) + \mu_A (L_A \cdot f(\cdot)) + \mu_M (L_M \cdot g(\cdot))$$

Along any interior optimal path, the following optimality conditions must hold:

$$u_c = N\mu_K \tag{159}$$

$$-u_h = \mu_K \frac{\partial Y}{\partial h} + \mu_A \frac{\partial \dot{A}}{\partial h} + \mu_M \frac{\partial \dot{M}}{\partial h} \tag{160}$$

$$\mu_K \frac{\partial Y}{\partial s_M} = \mu_M \frac{\partial \dot{M}}{\partial s_M} \tag{161}$$

$$\mu_K \frac{\partial Y}{\partial s_A} = \mu_A \frac{\partial \dot{A}}{\partial s_A} \tag{162}$$

$$\rho - \frac{\dot{\mu}_M}{\mu_M} = \frac{u_M}{\mu_M} \tag{163}$$

$$\rho - \frac{\dot{\mu}_A}{\mu_A} = \frac{\mu_K}{\mu_A}\frac{\partial Y}{\partial A} + \frac{\partial \dot{A}}{\partial A} + \frac{\mu_M}{\mu_A}\frac{\partial \dot{M}}{\partial A} \tag{164}$$

$$\rho - \frac{\dot{\mu}_K}{\mu_K} = \frac{\partial Y}{\partial K} + \frac{\mu_A}{\mu_K}\frac{\partial \dot{A}}{\partial K} + \frac{\mu_M}{\mu_K}\frac{\partial \dot{M}}{\partial K} - \delta \tag{165}$$

where μ_K, μ_A, μ_M are the costate variables associated with capital, traditional and leisure technology respectively. Combining these conditions, we get:

$$-\frac{u_h}{u_c} = \frac{1}{N}\frac{\partial Y}{\partial h} + \frac{\mu_A}{\mu_K}\frac{1}{N}\frac{\partial \dot{A}}{\partial h} + \frac{\mu_M}{\mu_K}\frac{1}{N}\frac{\partial \dot{M}}{\partial h} \tag{166}$$

$$\frac{u_M}{u_c} = \left(\rho - \frac{\dot{\mu}_M}{\mu_M}\right)\cdot\frac{1}{N}\frac{\mu_M}{\mu_K}. \tag{167}$$

$$\frac{\mu_M}{\mu_K} = \frac{\partial Y}{\partial s_M}\Big/\frac{\partial \dot{M}}{\partial s_M} \tag{168}$$

On a BGP all three terms on the right-hand side of (166) must grow at the same rate. Consider the first term, $\frac{1}{N}\frac{\partial Y}{\partial h}$. Since the marginal product of labor $\frac{\partial Y}{\partial L_Y}$, which grows at rate g_A, equals $\frac{\partial Y}{\partial h}\frac{\partial h}{\partial L_Y} = \frac{1}{s_Y N}\frac{\partial Y}{\partial h}$ and the share of labor employed in the Y-producing sector s_Y is constant on a BGP, the term $\frac{1}{N}\frac{\partial Y}{\partial h}$ grows at g_A. Thus we require all of the three terms on the right-hand side of (166) to grow at g_A. Consider the final term. Since $\frac{\partial \dot{M}}{\partial h} = \frac{\dot{M}}{h} = \frac{\dot{M}}{M}\frac{M}{h}$, the growth rate of $\frac{\partial \dot{M}}{\partial h}$ is $\frac{g_M}{g_h}$, implying that $\frac{\mu_M}{\mu_K}\frac{1}{N}$ grows at $\frac{g_h}{g_M}\cdot g_A$. Therefore, the right side of (167), and hence $\frac{u_M}{u_c}$, grow at $\frac{g_c}{g_M}$.

Note that equation (168) does not impose additional restrictions on the steady state growth rates. This is because we can write $\frac{\partial Y}{\partial s_M} = \frac{\partial Y}{\partial h}\frac{\partial h}{\partial L_Y}\frac{\partial L_Y}{\partial s_M} = -\frac{\partial Y}{\partial h}\frac{h}{(1-s_A-s_M)}$ and $\frac{\partial \dot{M}}{\partial s_M} = \frac{\partial \dot{M}}{\partial h}\frac{\partial h}{\partial L_M}\frac{\partial L_M}{\partial s_M} = \frac{\partial \dot{M}}{\partial h}\frac{h}{s_M}$, thus $\frac{\partial Y}{\partial s_M}/\frac{\partial \dot{M}}{\partial s_M}$ grows at the same rate as $\frac{\partial Y}{\partial h}/\frac{\partial \dot{M}}{\partial h}$, which is consistent with (166).

Similarly from (159) and (165) we get the Euler equation. It is useful to write the discrete time version as:

$$\frac{u_c(c,h,M)}{u_c(cg_c, hg_h, Mg_M)} = R. \tag{169}$$

for some constant R.

As shown in the decentralized equilibrium model of leisure technologies in Sections 4 and 5, the equilibrium conditions do not feature the marginal rate of substitution between leisure products and consumption goods $\frac{u_M}{u_c}$; thus for the equilibrium to feature a BGP, this MRS does not need to grow at $\frac{g_c}{g_M}$.

D.1.2 Definition of BGP preferences

We can now define the balanced growth preferences:

Definition 5. The utility function u is consistent with balanced growth in equilibrium with exponential growth in traditional technology and zero-price leisure technology if it is twice continuously differentiable and, given constants $h, c, M, g_A, g_M \zeta, \varrho > 0$, there exist

w and R such that:

$$-\frac{u_h(cg_c^t, hg_h^t, Mg_m^t)}{u_c(cg_c^t, hg_h^t, Mg_m^t)} = wg_A^t \tag{170}$$

$$\frac{u_c(cg_c^t, hg_h^t, Mg_m^t)}{u_c(cg_c^{t+1}, hg_h^{t+1}, Mg_m^{t+1})} = R \tag{171}$$

Such utility function is consistent with balanced growth in the planning problem if, in addition, there exists an m such that

$$\frac{u_M(cg_c^t, hg_h^t, Mg_m^t)}{u_c(cg_c^t, hg_h^t, Mg_m^t)} = m\frac{g_c^t}{g_M^t}. \tag{172}$$

Intuitively, the definition requires that when we scale variables arbitrarily but consistently with the BGP growth rates, the optimality conditions continue to hold. In the planning problem, all three of the (166), (167), (169) conditions must hold. In equilibrium with zero price leisure technologies only conditions (166) and (167) are required.

D.2 Proof of Proposition 2

The proof relies on 2 lemmas, which I now state and prove.

Define $\lambda = g_A^t$ and $\mu = g_M^t$ so that on the BGP:

$$-\frac{u_h(c\lambda^{1-\varrho}\mu^{-\zeta}, h\lambda^{-\varrho}\mu^{-\zeta}, M\mu)}{u_c(c\lambda^{1-\varrho}\mu^{-\zeta}, h\lambda^{-\varrho}\mu^{-\zeta}, M\mu)} = w\lambda \tag{173}$$

$$\frac{u_M(c\lambda^{1-\varrho}\mu^{-\zeta}, h\lambda^{-\varrho}\mu^{-\zeta}, M\mu)}{u_c(c\lambda^{1-\varrho}\mu^{-\zeta}, h\lambda^{-\varrho}\mu^{-\zeta}, M\mu)} = m\lambda^{1-\varrho}\mu^{-1-\zeta} \tag{174}$$

$$\frac{u_c(c\lambda^{1-\varrho}\mu^{-\zeta}, h\lambda^{-\varrho}\mu^{-\zeta}, M\mu)}{u_c(c\lambda^{1-\varrho}\mu^{-\zeta}g_A^{1-\varrho}g_M^{-\zeta}, h\lambda^{-\varrho}\mu^{-\zeta}g_A^{-\varrho}g_M^{-\zeta}, M\mu g_M)} = R. \tag{175}$$

Lemma 2. *If $u(c, h, M)$ satisfies (173) and 174 for all $\lambda > 0$ and $\mu > 0$, and for arbitrary $c > 0$, $w > 0$, $m > 0$, then its marginal rate of substitution functions, defined by $u_h(c, h, M)/u_c(c, h, M)$ and $u_M(c, h, M)/u_c(c, h, M)$ must be of the form*

$$\frac{u_h(c, h, M)}{u_c(c, h, M)} = \frac{c}{h} \cdot z\left(c^{\varrho}h^{1-\varrho}M^{\zeta}\right) \tag{176}$$

$$\frac{u_M(c, h, M)}{u_c(c, h, M)} = \frac{c}{M} \cdot y\left(c^{\varrho}h^{1-\varrho}M^{\zeta}\right) \tag{177}$$

for some functions z and y.

Proof. Since λ and μ are arbitrary, set $c\lambda^{1-\varrho}\mu^{-\zeta} = 1$ and $h\lambda^{-\varrho}\mu^{-\zeta} = 1$, implying $\lambda = \frac{h}{c}$ and $\mu = (c^{\varrho}h^{1-\varrho})^{\frac{1}{\zeta}}$. Separately also set $c\lambda^{1-\varrho}\mu^{-\zeta} = 1$ and $M\mu = 1$, implying $\mu = \frac{1}{M}$,

$\lambda = \left(\frac{M^{-\zeta}}{c}\right)^{\frac{1}{1-\varrho}}$ and $h\lambda^{-\varrho}\mu^{-\zeta} = hc^{\frac{\varrho}{1-\varrho}}M^{\zeta\frac{1}{1-\varrho}}$. Thus:

$$-\frac{u_h\left(1,1,M\left(c^{\varrho}h^{1-\varrho}\right)^{\frac{1}{\zeta}}\right)}{u_c\left(1,1,M\left(c^{\varrho}h^{1-\varrho}\right)^{\frac{1}{\zeta}}\right)} = w\frac{h}{c}$$
$$\frac{u_M\left(1,hc^{\frac{\varrho}{1-\varrho}}M^{\zeta\frac{1}{1-\varrho}},1\right)}{u_c\left(1,hc^{\frac{\varrho}{1-\varrho}}M^{\zeta\frac{1}{1-\varrho}},1\right)} = m\frac{M}{c}$$

Evaluating at $\lambda = 1$ and substituting the results for w and m we get:

$$\frac{u_h(c,h,M)}{u_c(c,h,M)} = \frac{c}{h}\frac{u_h\left(1,1,M\left(c^{\varrho}h^{1-\varrho}\right)^{\frac{1}{\zeta}}\right)}{u_c\left(1,1,M\left(c^{\varrho}h^{1-\varrho}\right)^{\frac{1}{\zeta}}\right)}$$
$$\frac{u_M(c,h,M)}{u_c(c,h,M)} = \frac{c}{M}\frac{u_M\left(1,hc^{\frac{\varrho}{1-\varrho}}M^{\zeta\frac{1}{1-\varrho}},1\right)}{u_c\left(1,hc^{\frac{\varrho}{1-\varrho}}M^{\zeta\frac{1}{1-\varrho}},1\right)}$$

Defining functions $z(t) = \frac{u_h(1,1,t^{1/\zeta})}{u_c(1,1,t^{1/\zeta})}$ and $y(t) = \frac{u_M(1,t^{\frac{1}{1-\varrho}},1)}{u_c(1,t^{\frac{1}{1-\varrho}},1)}$ yields the result. □

Lemma 3. *The second derivative of u must satisfy*

$$-\frac{u_{cc}c}{u_c} = p_1(c^{\varrho}h^{1-\varrho}M^{\zeta})$$
$$-\frac{u_{ch}c}{u_c} = p_2(c^{\varrho}h^{1-\varrho}M^{\zeta})$$
$$-\frac{u_{cM}c}{u_c} = p_3(c^{\varrho}h^{1-\varrho}M^{\zeta})$$

for some functions p_1, p_2, p_3.

Proof. Equation (169) can be written explicitly as a function of time:

$$\frac{u_c(cg_A^{t(1-\varrho)}g_M^{-t\zeta}, hg_A^{-t\varrho}\mu^{-\zeta}, M\mu)}{u_c(cg_A^{(t+1)(1-\varrho)}g_M^{-(t+1)\zeta}, hg_A^{-(t+1)\varrho}g_M^{-(t+1)\zeta}, Mg_M^{t+1})} = R \tag{178}$$

Differentiate (178) with respect to time, divide by (178) and set $t = 0$ to obtain:

$$\frac{u_{cc}(c,h,M)c\log\left(g_A^{1-\varrho}g_M^{-\zeta}\right) + u_{ch}(c,h,M)h\log\left(g_A^{-\varrho}g_M^{-\zeta}\right) + u_{cM}(c,h,M)M\log g_M}{u_c(c,h,M)} =$$

$$\frac{u_{cc}(cg_A^{1-\varrho}g_M^{-\zeta}, hg_A^{-\varrho}g_M^{-\zeta}, Mg_M)cg_A^{1-\varrho}g_M^{-\zeta}\log\left(g_A^{1-\varrho}g_M^{-\zeta}\right)}{u_c(cg_A^{1-\varrho}g_M^{-\zeta}, hg_A^{-\varrho}g_M^{-\zeta}, Mg_M)}$$

$$+\frac{u_{ch}(cg_A^{1-\varrho}g_M^{-\zeta}, hg_A^{-\varrho}g_M^{-\zeta}, Mg_M)hg_A^{-\varrho}g_M^{-\zeta}\log\left(g_A^{-\varrho}g_M^{-\zeta}\right)}{u_c(cg_A^{1-\varrho}g_M^{-\zeta}, hg_A^{-\varrho}g_M^{-\zeta}, Mg_M)}$$

$$+\frac{u_{cM}(cg_A^{1-\varrho}g_M^{-\zeta}, hg_A^{-\varrho}g_M^{-\zeta}, Mg_M)Mg_M\log\left(g_M\right)}{u_c(cg_A^{1-\varrho}g_M^{-\zeta}, hg_A^{-\varrho}g_M^{-\zeta}, Mg_M)} \quad (179)$$

Differentiating (176) and (177) with respect to c we see that $h\frac{u_{ch}}{u_c}$ and $M\frac{u_{cM}}{u_c}$ are functions of $c^\varrho h^{1-\varrho}M^\zeta$ and $\frac{u_{cc}c}{u_c}$ only. Thus we have:

$$hg_A^{-\varrho}g_M^{-\zeta}\frac{u_{ch}(cg_A^{1-\varrho}g_M^{-\zeta}, hg_A^{-\varrho}g_M^{-\zeta}, Mg_M)}{u_c(cg_A^{1-\varrho}g_M^{-\zeta}, hg_A^{-\varrho}g_M^{-\zeta}, Mg_M)} = f_1\left(c^\varrho h^{1-\varrho}M^\zeta, \frac{u_{cc}(cg_A^{1-\varrho}g_M^{-\zeta}, hg_A^{-\varrho}g_M^{-\zeta}, Mg_M)}{u_c(cg_A^{1-\varrho}g_M^{-\zeta}, hg_A^{-\varrho}g_M^{-\zeta}, Mg_M)}cg_A^{1-\varrho}g_M^{-\zeta}\right)$$

$$Mg_M\frac{u_{ch}(cg_A^{1-\varrho}g_M^{-\zeta}, hg_A^{-\varrho}g_M^{-\zeta}, Mg_M)}{u_c(cg_A^{1-\varrho}g_M^{-\zeta}, hg_A^{-\varrho}g_M^{-\zeta}, Mg_M)} = f_2\left(c^\varrho h^{1-\varrho}M^\zeta, \frac{u_{cc}(cg_A^{1-\varrho}g_M^{-\zeta}, hg_A^{-\varrho}g_M^{-\zeta}, Mg_M)}{u_c(cg_A^{1-\varrho}g_M^{-\zeta}, hg_A^{-\varrho}g_M^{-\zeta}, Mg_M)}cg_A^{1-\varrho}g_M^{-\zeta}\right)$$

Using these in equation (179) we get:

$$\frac{u_{cc}(c,h,M)c}{u_c(c,h,M)} = f_3\left(c^\varrho h^{1-\varrho}M^\zeta, \frac{u_{cc}(cg_A^{1-\varrho}g_M^{-\zeta}, hg_A^{-\varrho}g_M^{-\zeta}, Mg_M)}{u_c(cg_A^{1-\varrho}g_M^{-\zeta}, hg_A^{-\varrho}g_M^{-\zeta}, Mg_M)}cg_A^{1-\varrho}g_M^{-\zeta}\right)$$

This equation holds for any g_A and g_M. We can thus set $g_A = g_M = 1$ and conclude that $\frac{u_{cc}c}{u_c}$ only depends on $c^\varrho h^{1-\varrho}M^\zeta$.

We now turn to the proof of Theorem 1 for the case of optimal allocation, starting with the "only if" direction of the claim. Recall that $x := c^\varrho h^{1-\varrho}M^\zeta$. □

Case with $\varrho \neq 0$

Note that:

$$\frac{\partial\log u_{cc}}{\partial\log c} = \frac{\partial\log u_c}{\partial c}\frac{\partial c}{\partial\log c} = \frac{\partial\log u_c}{\partial c}\frac{1}{\frac{\partial\log c}{\partial c}} = \frac{\partial\log u_c}{\partial c}c = \frac{1}{u_c}u_{cc}c = \frac{u_{cc}(c,h)}{u_c(c,h)}c$$

so that, using Lemma 3:

$$\frac{\partial\log u_c}{\partial\log c} = -p\left(\exp\left(\varrho\log c + (1-\varrho)\log h + \zeta\log M\right)\right)$$

for some function p. Integrate this equation with respect to $\log c$. The result follows from the fact that derivative of $f(\exp(\log(x(\log(c)))))+m(h,M)$ is $f'(\exp(\log(x)))\cdot\exp(\log(x))\cdot \varrho = p(\exp(\log(x)))$ so that

$$\log u_c = f_2(x) + m_0(h,M) \tag{180}$$

Exponentiating:

$$u_c = f_3(x)m_1(h,M) \tag{181}$$

From the proof of Lemma we know that hu_{ch}/u_c and Mu_{cM}/u_c are only functions of x. Differentiating (181) with respect to h, multiplying by h and dividing through by u_c, we get:

$$\begin{aligned} u_{ch} &= f_{3'}(x)\frac{x}{h}m_1 + f_3(x)m_{1h} \\ hu_{ch} &= f_{3'}(x)xm_1 + f_3(x)m_{1h}h \\ \frac{hu_{ch}}{u_c} &= f_{3''}(x) + \frac{m_{1h}h}{m_1} \end{aligned}$$

Similarly, $Mu_{cM}/u_c = f_{4'}(x) + \frac{m_{1M}(h,M)\cdot M}{m_1(h,M)}$. Note the final terms in these equations do not depend on c. But these must either depend on x or be a constant, because we know this object depends only on x. Thus these terms are constant and it follows that m_1 is isoelastic:

$$m_1 = A_2 h^{\kappa} M^{\iota}$$

Thus

$$u_c = f_3(c^{\varrho}h^{1-\varrho}M^{\zeta})A_2 h^{\kappa}M^{\iota}. \tag{182}$$

Since $\varrho \neq 0$, we can write this as

$$u_c = f_5(ch^{\frac{1-\varrho}{\varrho}}M^{\frac{\zeta}{\varrho}})h^{\kappa}M^{\iota}. \tag{183}$$

Integrating with respect to c we get

$$u = f_6(ch^{\frac{1-\varrho}{\varrho}}M^{\frac{\zeta}{\varrho}})h^{\kappa-\frac{1-\varrho}{\varrho}}M^{\iota-\frac{\zeta}{\varrho}} + m_2(h,M) \tag{184}$$

where m_2 is a function of h and M only as c was integrated over.

Now use the result in Lemma 2 and equation (183) to get

$$u_h(c,h,M) = \frac{c}{h}\cdot z(x)\cdot u\_\{c\}(c,h,M) = A_2 c\cdot z(x)\cdot f_3(x)h^{\kappa-1}M^{\iota} \tag{185}$$

$$u_M(c,h,M) = \frac{c}{M}\cdot y(x)\cdot u\_\{c\}(c,h,M) = A_2 c\cdot y(x)\cdot f_3(x)h^{\kappa}M^{\iota-1} \tag{186}$$

Note that $x^{\frac{1}{\varrho}} = ch^{\frac{1-\varrho}{\varrho}} M^{\frac{\zeta}{\varrho}}$. Let $f_7(x) := A_2 x^{\frac{1}{\varrho}} z(x) f_3(x)$ so that

$$u_h = f_7(x) h^{\kappa - 1 - \frac{1-\varrho}{\varrho}} M^{\iota - \frac{\zeta}{\varrho}}$$

$$u_M = f_7(x) h^{\kappa - \frac{1-\varrho}{\varrho}} M^{\iota - 1 - \frac{\zeta}{\varrho}}$$

Now we take the derivatives of u directly, using (184):

$$u_h = f_{10}\left(ch^{\frac{1-\varrho}{\varrho}} M^{\frac{\zeta}{\varrho}}\right) h^{\kappa - 1 - \frac{1-\varrho}{\varrho}} M^{\iota - \frac{\zeta}{\varrho}} + m_{2,h}(h, M)$$
$$u_M = f_{11}\left(ch^{\frac{1-\varrho}{\varrho}} M^{\frac{\zeta}{\varrho}}\right) h^{\kappa - \frac{1-\varrho}{\varrho}} M^{\iota - 1 - \frac{\zeta}{\varrho}} + m_{2,M}(h, M)$$

For these two equations to be consistent with equations (185) and (186) for all c, h, M it must be that

$$m_{2,h}(h, M) = A_3 h^{\kappa - 1 - \frac{1-\varrho}{\varrho}} M^{\iota - \frac{\zeta}{\varrho}}$$

and

$$m_{2,M}(h, M) = A_4 h^{\kappa - \frac{1-\varrho}{\varrho}} M^{\iota - 1 - \frac{\zeta}{\varrho}}$$

where $A_3$and A_4 are constants, because $m_{2,h}$ and $m_{2,M}$ do not depend on c or x. Now we can integrate these functions to find m_2.

Consider first the case where $\kappa \neq \frac{1-\varrho}{\varrho}$ and $\iota \neq \frac{\zeta}{\varrho}$. Integrating we get:

$$m_2 = A_5 h^{\kappa - \frac{1-\varrho}{\varrho}} M^{\iota - \frac{\zeta}{\varrho}} + g_1(M)$$
$$m_2 = A_6 h^{\kappa - \frac{1-\varrho}{\varrho}} M^{\iota - \frac{\zeta}{\varrho}} + g_2(h)$$

Together these imply $A_5 = A_6$ and $g_1 = g_2 = constant$. Using this in (184) we thus get

$$u = f_{12}(ch^{\frac{1-\varrho}{\varrho}} M^{\frac{\zeta}{\varrho}}) h^{\kappa - \frac{1-\varrho}{\varrho}} M^{\iota - \frac{\zeta}{\varrho}} + A_7$$

Note that $x = c^{\varrho} h^{1-\varrho} M^{\zeta}$ and thus $h^{\kappa - \frac{1-\varrho}{\varrho}} M^{\iota - \frac{\zeta}{\varrho}} = \left(\frac{x}{c^{\varrho}}\right)^{-\frac{1}{\varrho}} h^{\kappa} M^{\iota}$ and so we can write:

$$u = f_{12}(ch^{\frac{1-\varrho}{\varrho}} M^{\frac{\zeta}{\varrho}}) \left(\frac{x}{c^{\varrho}}\right)^{-\frac{1}{\varrho}} h^{\kappa} M^{\iota} + A_7$$

which is

$$u = f_{13}(ch^{\frac{1-\varrho}{\varrho}} M^{\frac{\zeta}{\varrho}}) \cdot ch^{\kappa} M^{\iota} + A_7$$

Since f_{13} is a function of $x^{1/\varrho}$ it can be written as a function of x:

$$u = f_{14}(c^{\varrho} h^{1-\varrho} M^{\zeta}) \cdot ch^{\kappa} M^{\iota} + A_7$$

Now again $x = c^{\varrho} h^{1-\varrho} M^{\zeta}$ so $h^{\kappa} = \left(\frac{x}{c^{\varrho} M^{\zeta}}\right)^{\kappa/1-\varrho}$ so this can be written as

$$u = f_{15}(c^{\varrho} h^{1-\varrho} M^{\zeta}) \cdot c^{1-\kappa \frac{\varrho}{1-\varrho}} M^{\iota - \frac{\kappa\zeta}{1-\varrho}} + A_7$$

That in turn is equivalent to

$$u = \frac{\left(c^{\epsilon} M^{1-\epsilon} v(c^{\varrho} h^{1-\varrho} M^{\zeta})\right)^{1-\sigma} - 1}{1-\sigma}$$

where we have set $A_7 = -\frac{1}{1-\sigma}$, $\epsilon(1-\sigma) = 1 - \kappa \frac{\varrho}{1-\varrho}$, $(1-\sigma)(1-\epsilon) = \iota - \kappa \frac{\zeta}{1-\varrho}$.

Consider now the case with $\kappa = \frac{1-\varrho}{\varrho}$ and $\iota = \frac{\zeta}{\varrho}$. Integrating we get that

$$m_2(h, M) = A_8 \log h + A_9 \log M + A_{10}$$

So that we can write (184) as

$$u = f_5(c h^{\frac{1-\varrho}{\varrho}} M^{\frac{\zeta}{\varrho}}) h^{\kappa - \frac{1-\varrho}{\varrho}} M^{\iota - \frac{\zeta}{\varrho}} + A_8 \log h + A_9 \log M + A_{10}$$

Note that we can eliminate $\log h$ because we know that

$$\log(c h^{\frac{1-\varrho}{\varrho}} M^{\frac{\zeta}{\varrho}}) = \log c + \frac{1-\varrho}{\varrho} \log h + \frac{\zeta}{\varrho} \log M$$

thus

$$u = f_{16}(c^{\varrho} h^{1-\varrho} M^{\zeta}) + A_{11} \log c + A_{12} \log M.$$

Normalize the constants A_{11} and A_{12} to sum to 1 to get:

$$u = \epsilon \log c + (1-\epsilon) \log M + \log v(c^{\varrho} h^{1-\varrho} M^{\zeta}).$$

Taken together, the two cases imply that the utility function takes the form

$$u(c, h, M) = \begin{cases} \frac{\left(c^{\epsilon} M^{1-\epsilon} v\left(c^{\varrho} h^{1-\varrho} M^{\zeta}\right)\right)^{1-\sigma}}{1-\sigma} & \sigma \neq 1 \\ \log\left(c^{\epsilon} M^{1-\epsilon}\right) + \log v(c^{\varrho} h^{1-\varrho} M^{\zeta}) & \sigma = 1 \end{cases}.$$

Case with $\varrho = 0$.

Now $x = h M^{\zeta}$ and so:

$$\frac{\partial \log u_c}{\partial \log c} = -p\left(\exp\left(\log h + \zeta \log M\right)\right)$$

Integrate with respect to $\log c$. In this case it's simple:

$$\log u_c = -z(x)\log(c) + m_3(h, M) \tag{187}$$

From the proof of Lemma 2 we know that hu_{ch}/u_c and Mu_{cM}/u_c are only functions of x. Differentiate the last equation wrt h and multiply through by h:

$$hu_{ch}/u_c = \log(c)z'(x)hM^{\zeta} + hm_{3,h}(h, M)$$

Thus $z'(x)$ must be zero. Similarly:

$$Mu_{cM}/u_c = \log(c)z'(x)\zeta hM^{\zeta} + Mm_{3,M}(h, M)$$

If $z'(x)$ is zero, $z(x)$ is a constant. Let $z(x) = \sigma$. Then equation (187) reads:

$$\log u_c = \log(c^{-\sigma}) + m_3(h, M)$$

Differentiating with respect to h:

$$\frac{hu_{ch}}{u_c} = hm_{3,h}$$

Thus it follows from Lemma 2 that $hm_{3,h}$ is a function of x only:

$$hm_{3,h} = g_1(x).$$

Integrating, we get that $m_3 = F_1(x) + G_1(M)$, where G depends on M only as h was integrated over.

Differentiating with respect to M:

$$\frac{Mu_{cM}}{u_c} = Mm_{3,M}$$

thus

$$Mm_{3M} = g_2(x)$$

Integrating, we get that $m_3 = F_2(x) + G_2(h)$, where G depends on M only as h was integrated over. For both of these to be true, $G_1 = G_2$ are constants and so m_3 is a function of x only.

Exponentiating (187) we get:

$$u_c = c^{-\sigma}m_4(x) \tag{188}$$

Suppose $\sigma \neq 1$. Integrating with respect to c we find:

$$u = \frac{c^{1-\sigma}}{1-\sigma} m_4(x) + m_5(h, M)$$

Which can be written as

$$u = \frac{\left(cv(hM^{\zeta})\right)^{1-\sigma}}{1-\sigma} + m_5(h, M) \tag{189}$$

From Lemma (2) we have:

$$\begin{aligned} u_h(c, h, M) =& \frac{c}{h} \cdot z\left(c^{\varrho} h^{1-\varrho} M^{\zeta}\right) u_c(c, h, M) \\ u_M(c, h, M) =& \frac{c}{M} \cdot y\left(c^{\varrho} h^{1-\varrho} M^{\zeta}\right) u_c(c, h, M) \end{aligned}$$

Combining with (188):

$$\begin{aligned} u_h(c, h, M) =& \frac{1}{h} \cdot z\left(hM^{\zeta}\right) c^{1-\sigma} m_4(hM^{\zeta}) \\ u_M(c, h, M) =& \frac{1}{M} \cdot y\left(hM^{\zeta}\right) c^{1-\sigma} m_4(hM^{\zeta}) \end{aligned}$$

Differentiating (189) with respect to h and M yields:

$$\begin{aligned} u_h &= \left(cv(hM^{\zeta})\right)^{-\sigma} cv'(hM^{\zeta}) hM^{\zeta} \frac{1}{h} + m_{5,h} \\ u_M &= \left(cv(hM^{\zeta})\right)^{-\sigma} cv'(hM^{\zeta}) hM^{\zeta} \frac{1}{M} + m_{5,M} \end{aligned}$$

Comparing the last two pairs of equations, m_5 must be a constant that can be ignored. Thus in this case:

$$u(c, h, M) = \frac{\left(cv(hM^{\zeta})\right)^{1-\sigma}}{1-\sigma}$$

In the case with $\sigma = 1$, we have

$$u_c = \frac{1}{c} m_4(x) \tag{190}$$

Integrating with respect to c, we get that

$$u = \log c \cdot m_4(x) + m_6(h, M) \tag{191}$$

Again, Lemma 1 combined with equation (190) yields:

$$u_h(c,h,M) = \frac{1}{h} \cdot z\left(hM^\zeta\right) m_4(hM^\zeta)$$
$$u_M(c,h,M) = \frac{1}{M} \cdot y\left(hM^\zeta\right) m_4(hM^\zeta)$$

Differentiating (191) with respect to h and with respect to M:

$$u_h(c,h,M) = \log c \cdot m_4'(x) \cdot \frac{x}{h} + m_{6,h}(h,M)$$
$$u_M(c,h,M) = \log c \cdot m_4'(x) \cdot \frac{x}{M} + m_{6,M}(h,M)$$

Comparing the last two sets of equations we conclude that $m_4'(x)$ must be zero (therefore m_4 is a constant) and that

$$m_{6,h}(h,M) = \frac{1}{h} \cdot m_7(x)$$
$$m_{6,M}(h,M) = \frac{1}{M} \cdot m_8(x)$$

Integrating, we get that, first, $m_6 = F(x) + G(M)$ and, second, $m_6 = FF(x) + GG(h)$. Therefore G and GG are constants that are equal and m_6 depends only on x. Altogether, we have:

$$u = \log c + m_6(hM^\zeta).$$

We thus conclude that in the case when $\varrho = 0$ the utility function is of the form:

$$u(c,h,M) = \begin{cases} \frac{\left(cv(hM^\zeta)\right)^{1-\sigma}}{1-\sigma} & \sigma \neq 1 \\ \log c + \log v(hM^\zeta) & \sigma = 1 \end{cases}.$$

D.3 Sufficiency: optimal allocation

To verify sufficiency take the derivatives of the utility function with respect to its three arguments:

$$\frac{\left(c^\epsilon M^{1-\epsilon} v\left(c^\varrho h^{1-\varrho} M^\zeta\right)\right)^{1-\sigma}}{1-\sigma}$$

$$u_c(c,h,M) = \left(c^{\epsilon}M^{1-\epsilon}v\left(x\right)\right)^{-\sigma}\left(\epsilon c^{\epsilon-1}M^{1-\epsilon}v(x) + c^{\epsilon}M^{1-\epsilon}v'(x)\varrho\frac{x}{c}\right)$$
$$u_h(c,h,M) = \left(c^{\epsilon}M^{1-\epsilon}v\left(x\right)\right)^{-\sigma}\left(c^{\epsilon}M^{1-\epsilon}v'(x)(1-\varrho)\frac{x}{h}\right)$$
$$u_M(c,h,M) = \left(c^{\epsilon}M^{1-\epsilon}v\left(x\right)\right)^{-\sigma}\left((1-\epsilon)c^{\epsilon}M^{-\epsilon}v(x) + c^{\epsilon}M^{1-\epsilon}v'(x)\zeta\frac{x}{M}\right)$$

Combining these we get that

$$\frac{u_h(c,h,M)}{u_c(c,h,M)} = \frac{c^{\epsilon}M^{1-\epsilon}v'(x)(1-\varrho)\frac{x}{h}}{\epsilon c^{\epsilon-1}M^{1-\epsilon}v(x) + c^{\epsilon}M^{1-\epsilon}v'(x)\varrho\frac{x}{c}} = \frac{c}{h}\frac{v'(x)(1-\varrho)x}{\epsilon v(x) + v'(x)x\varrho}$$

Multiplying c by $\lambda^{1-\varrho}\mu^{-\zeta}$, h by $\lambda^{-\varrho}\mu^{-\zeta}$ we find that this expression increases by a factor of λ, thus verifying (173). Similarly, we have

$$\frac{u_M(c,h,M)}{u_c(c,h,M)} = \frac{(1-\epsilon)c^{\epsilon}M^{-\epsilon}v(x) + c^{\epsilon}M^{1-\epsilon}v'(x)\zeta\frac{x}{M}}{\epsilon c^{\epsilon-1}M^{1-\epsilon}v(x) + c^{\epsilon}M^{1-\epsilon}v'(x)\varrho\frac{x}{c}} = \frac{c}{M}\frac{(1-\epsilon)v(x) + v'(x)\zeta x}{\epsilon v(x) + v'(x)\varrho x}.$$

Multiplying c by $\lambda^{1-\varrho}\mu^{-\zeta}$, M by μ we find that this expression increases by a factor of $\lambda^{1-\varrho}\mu^{-1-\zeta}$, thus verifying (174). Finally, evaluate

$$\frac{u_c(c,h,M)}{u_c(cg_A^{1-\varrho}g_M^{-\zeta}, hg_A^{-\varrho}g_M^{-\zeta}, Mg_M)} = g_A^{(1-\varrho)(1-\epsilon(1-\sigma))}g_M^{\zeta((1-\varrho)(\epsilon(1-\sigma)-1)+(1-\epsilon)(1-\sigma))}$$

which is independent of c, h and M. Letting $R := g_A^{(1-\varrho)(1-\epsilon(1-\sigma))}g_M^{\zeta((1-\varrho)(\epsilon(1-\sigma)-1)+(1-\epsilon)(1-\sigma))}$ we see that (169) is also satisfied.

D.4 BGP preferences in the decentralized equilibrium

In a competitive equilibrium with zero price technologies, the MRS functions must still obey (173) and (169), but not (174). Thus analysis above implies that equation (176) in Lemma 2 and all equations in 3 continue to hold.

D.4.1 The case with $\varrho = 0$.

In this case $x = hM^{\zeta}$ and so Lemma 3 implies:

$$\frac{\partial \log u_c}{\partial \log c} = -p\left(\exp\left(\log h + \zeta \log M\right)\right)$$

Integrate with respect to $\log c$:

$$\log u_c = -z(x)\log(c) + m_3(h,M) \tag{192}$$

Differentiate equation (192) wrt h and multiply through by h, and similarly with M:

$$
\begin{aligned}
hu_{ch}/u_c =& \log(c)z'(x)hM^\zeta + hm_{3,h}(h,M) \\
Mu_{cM}/u_c =& \zeta\log(c)z'(x)hM^\zeta + Mm_{3,M}(h,M)
\end{aligned}
$$

From the proof of Lemma 3 we know that hu_{ch}/u_c and Mu_{cM}/u_c are only functions of x. Thus $z'(x)$ must be zero. If $z'(x)$ is zero, $z(x)$ is a constant. Let $z(x) = \sigma$. Then equation (192) reads:

$$
\log u_c = \log(c^{-\sigma}) + m_3(h,M)
$$

and

$$
\begin{aligned}
hu_{ch}/u_c &= hm_{3,h}(h,M) \\
Mu_{cM}/u_c &= Mm_{3,M}(h,M)
\end{aligned}
$$

Thus it follows from Lemma 3 that $hm_{3,h}$ and $Mm_{3,M}$ are functions of x only and so:

$$
\begin{aligned}
m_{3,h} =& \frac{1}{h}g_1(x) \\
m_{3,M} =& \frac{1}{M}g_2(x)
\end{aligned}
$$

Integrating we get that:

$$
\begin{aligned}
m_3(h,M) =& g_4(x) + g_5(M) \\
m_3(h,M) =& g_6(x) + g_7(h)
\end{aligned}
$$

This implies that g_5 and g_7 are identical constants so m_3 depends only on x. Thus:

$$
\log u_c = \log(c^{-\sigma}) + g_4(x)
$$

Exponentiating, we get:

$$
u_c = c^{-\sigma}m_4(x) \tag{193}
$$

Now we need to consider two cases separately.

Case with $\sigma \neq 1$. Integrating with respect to c we find:

$$
u = \frac{c^{1-\sigma}}{1-\sigma}m_4(x) + m_6(h,M)
$$

Which can be written as

$$u = \frac{\left(cv(hM^{\zeta})\right)^{1-\sigma}}{1-\sigma} + m_6(h, M) \tag{194}$$

From equation (176) in Lemma 2 we have:

$$u_h(c, h, M) = \frac{c}{h} \cdot z\left(hM^{\zeta}\right) u_c(c, h, M)$$

Combining with (193):

$$u_h(c, h, M) = \frac{c^{1-\sigma}}{h} \cdot z\left(hM^{\zeta}\right) m_4(hM^{\zeta})$$

Now, differentiating (194) with respect to h:

$$u_h = \frac{c^{1-\sigma}}{h} \left(v(hM^{\zeta})\right)^{-\sigma} v'(hM^{\zeta})hM^{\zeta} + m_{6,h}$$

Comparing the last two equations, $m_{6,h}$ must be zero, so m_6 depends only on M. Thus

$$u = \frac{\left(cv(hM^{\zeta})\right)^{1-\sigma}}{1-\sigma} + f(M).$$

Case with $\sigma = 1$. Equation (193) becomes:

$$u_c = \frac{1}{c} m_4(x)$$

Integrating with respect to c we find:

$$u = \log c \cdot m_4(x) + m_6(h, M) \tag{195}$$

From Lemma 2 we have:

$$u_h(c, h, M) = \frac{c}{h} \cdot z\left(hM^{\zeta}\right) u_c(c, h, M)$$

Combining with (193):

$$u_h(c, h, M) = \frac{1}{h} \cdot z\left(hM^{\zeta}\right) m_4(hM^{\zeta})$$

Differentiating (195) with respect to h:

$$u_h(c,h,M) = \log c \cdot m_4'(x) \cdot \frac{x}{h} + m_{6,h}(h,M)$$

Comparing the last two equations, we must have m_4 being a constant, in that case

$$u = \log c + m_6(h,M)$$

where m_6 satisfies

$$m_{6,h}(h,M) = \frac{1}{h} \cdot m_7(x)$$

Integrating we must have

$$m_6 = m_8(x) + m_6(M)$$

Thus, appropriately defining v as $\exp m_8$, we finally obtain:

$$u = \log c + \log v(hM^{\zeta}) + f(M)$$

D.4.2 Case with $\varrho \neq 0$

Note that:

$$\frac{\partial \log u_c}{\partial \log c} = \frac{\partial \log u_c}{\partial c}\frac{\partial c}{\partial \log c} = \frac{\partial \log u_c}{\partial c}\frac{1}{\frac{\partial \log c}{\partial c}} = \frac{\partial \log u_c}{\partial c}c = \frac{1}{u_c}u_{cc}c = \frac{u_{cc}(c,h)}{u_c(c,h)}c$$

so that, using Lemma 3:

$$\frac{\partial \log u_c}{\partial \log c} = -p\left(\exp\left(\varrho \log c + (1-\varrho)\log h + \zeta \log M\right)\right)$$

for some function p. Integrate this equation with respect to $\log c$:

$$\log u_c = f_2(x) + m_0(h,M) \tag{196}$$

This result follows from the fact that derivative of $f(\exp(\log(x(\log(c))))) + m(h,M)$ is $f'(\exp(\log(x))) \cdot \exp(\log(x)) \cdot \varrho = p(\exp(\log(x)))$. Exponentiating:

$$u_c = f_3(x)m_1(h,M) \tag{197}$$

From the proof of Lemma 2 we know that hu_{ch}/u_c and Mu_{cM}/u_c are only functions of x. Differentiating (197) with respect to h, multiplying by h and dividing through by

u_c, we get:

$$u_{ch} = f_{3'}(x)\frac{x}{h}m_1 + f_3(x)m_{1h}$$
$$hu_{ch} = f_{3'}(x)xm_1 + f_3(x)m_{1h}h$$
$$\frac{hu_{ch}}{u_c} = f_{3''}(x) + \frac{m_{1h}h}{m_1}$$

Similarly, $Mu_{cM}/u_c = f_{4'}(x) + \frac{m_{1,M}\cdot M}{m_1}$. Note the final terms in these equations do not depend on c. But these must either depend on x or be a constant, because we know this object depends only on x. Thus these terms are constant and it follows that m_1 is isoelastic:

$$m_1 = A_2 h^{\kappa} M^{\iota}$$

Thus

$$u_c = f_3(c^{\varrho}h^{1-\varrho}M^{\zeta})A_2h^{\kappa}M^{\iota}. \tag{198}$$

Since $\varrho \neq 0$, we can write this as

$$u_c = f_5(ch^{\frac{1-\varrho}{\varrho}}M^{\frac{\zeta}{\varrho}})A_2h^{\kappa}M^{\iota}. \tag{199}$$

Integrating with respect to c we get

$$u = f_6(ch^{\frac{1-\varrho}{\varrho}}M^{\frac{\zeta}{\varrho}})h^{\kappa-\frac{1-\varrho}{\varrho}}M^{\iota-\frac{\zeta}{\varrho}} + m_2(h, M) \tag{200}$$

where m_2 is a function of h and M only as c was integrated over.

Now use equation (176) in Lemma 2 and (199):

$$u_h(c, h, M) = \frac{c}{h}\cdot z(x)\cdot u_c(c, h, M) = A_2c\cdot z(x)\cdot f_3(x)h^{\kappa-1}M^{\iota} \tag{201}$$

Note that $\tilde{x} := x^{\frac{1}{\varrho}} = ch^{\frac{1-\varrho}{\varrho}}M^{\frac{\zeta}{\varrho}}$. Let $f_7(x) := A_2x^{\frac{1}{\varrho}}z(x)f_3(x)$ so that

$$u_h = f_7(x)h^{\kappa-1-\frac{1-\varrho}{\varrho}}M^{\iota-\frac{\zeta}{\varrho}} \tag{202}$$

Now we take the derivative of u with respect to h directly, using (200):

$$u_h = f_{10}(\tilde{x})\frac{\tilde{x}}{h}h^{\kappa-\frac{1-\varrho}{\varrho}}M^{\iota-\frac{\zeta}{\varrho}} + (\kappa - \frac{1-\varrho}{\varrho})f_6(\tilde{x})h^{\kappa-1-\frac{1-\varrho}{\varrho}}M^{\iota-\frac{\zeta}{\varrho}} + m_{2,h}(h, M).$$

This yields:

$$u_h = f_{10}\left(ch^{\frac{1-\varrho}{\varrho}}M^{\frac{\zeta}{\varrho}}\right)h^{\kappa-1-\frac{1-\varrho}{\varrho}}M^{\iota-\frac{\zeta}{\varrho}} + m_{2,h}(h, M)$$

which can be written instead as a function of x:

$$u_h = f_{11}\,(x)\,h^{\kappa-1-\frac{1-\varrho}{\varrho}}M^{\iota-\frac{\zeta}{\varrho}} + m_{2,h}(h,M) \tag{203}$$

For equations (202) and (203) to be consistent for all c, h, M it must be that

$$m_{2,h}(h,M) = A_3 h^{\kappa-1-\frac{1-\varrho}{\varrho}}M^{\iota-\frac{\zeta}{\varrho}} \tag{204}$$

because m_{2h} does not depend on c (or x). Now we can integrate this function to find m_2.

Consider first the case where $\kappa \neq \frac{1-\varrho}{\varrho}$ and $\iota \neq \frac{\zeta}{\varrho}$. Integrating we get:

$$m_2 = A_5 h^{\kappa-\frac{1-\varrho}{\varrho}}M^{\iota-\frac{\zeta}{\varrho}} + g_1(M)$$

Using this in (200) we thus get

$$u = f_6(ch^{\frac{1-\varrho}{\varrho}}M^{\frac{\zeta}{\varrho}})h^{\kappa-\frac{1-\varrho}{\varrho}}M^{\iota-\frac{\zeta}{\varrho}} + m_2(h,M) = f_{12}(ch^{\frac{1-\varrho}{\varrho}}M^{\frac{\zeta}{\varrho}})h^{\kappa-\frac{1-\varrho}{\varrho}}M^{\iota-\frac{\zeta}{\varrho}} + g_1(M)$$

Note that $x = c^\varrho h^{1-\varrho}M^\zeta$ and thus $h^{\kappa-\frac{1-\varrho}{\varrho}}M^{\iota-\frac{\zeta}{\varrho}} = \left(\frac{x}{c^\varrho}\right)^{-\frac{1}{\varrho}} h^\kappa M^\iota$ and so we can write:

$$u = f_{12}(ch^{\frac{1-\varrho}{\varrho}}M^{\frac{\zeta}{\varrho}})\left(\frac{x}{c^\varrho}\right)^{-\frac{1}{\varrho}} h^\kappa M^\iota + g_1(M)$$

which is

$$u = f_{13}(ch^{\frac{1-\varrho}{\varrho}}M^{\frac{\zeta}{\varrho}}) \cdot ch^\kappa M^\iota + g_1(M)$$

Since f_{13} is a function of $x^{1/\varrho}$ it can be written as a function of x:

$$u = f_{14}(c^\varrho h^{1-\varrho}M^\zeta) \cdot ch^\kappa M^\iota + g_1(M)$$

Now again $x = c^\varrho h^{1-\varrho}M^\zeta$ so $h^\kappa = \left(\frac{x}{c^\varrho M^\zeta}\right)^{\kappa/1-\varrho}$ so this can be written as

$$u = f_{15}(c^\varrho h^{1-\varrho}M^\zeta) \cdot c^{1-\kappa\frac{\varrho}{1-\varrho}}M^{\iota-\frac{\kappa\zeta}{1-\varrho}} + g_1(M)$$

That in turn is equivalent to

$$u = \frac{\left(c^\epsilon M^{1-\epsilon}v(c^\varrho h^{1-\varrho}M^\zeta)\right)^{1-\sigma}}{1-\sigma} + f(M)$$

where we have set $\epsilon(1-\sigma) = 1-\kappa\frac{\varrho}{1-\varrho}$, $(1-\sigma)(1-\epsilon) = \iota - \kappa\frac{\zeta}{1-\varrho}$.

Consider now the case with $\kappa = \frac{1-\varrho}{\varrho}$ and $\iota = \frac{\zeta}{\varrho}$. Integrating (204) (which is $m_{2,1}(h,M) =$

$A_3\frac{1}{h}$) we get that

$$m_2(h,M) = A_8 \log h + F(M)$$

So that we can write (200) as

$$u = f_5(ch^{\frac{1-\varrho}{\varrho}} M^{\frac{\zeta}{\varrho}}) h^{\kappa - \frac{1-\varrho}{\varrho}} M^{\iota - \frac{\zeta}{\varrho}} + A_8 \log h + F(M)$$

Note that $x = c^{\varrho} h^{1-\varrho} M^{\zeta}$ and thus $h^{\kappa - \frac{1-\varrho}{\varrho}} M^{\iota - \frac{\zeta}{\varrho}} = \left(\frac{x}{c^{\varrho}}\right)^{-\frac{1}{\varrho}} h^{\kappa} M^{\iota}$ and so we can write:

$$u = f_5(ch^{\frac{1-\varrho}{\varrho}} M^{\frac{\zeta}{\varrho}}) \left(\frac{x}{c^{\varrho}}\right)^{-\frac{1}{\varrho}} h^{\kappa} M^{\iota} + A_8 \log h + F(M)$$

Note that we can eliminate $\log h$ because we know that

$$\log(ch^{\frac{1-\varrho}{\varrho}} M^{\frac{\zeta}{\varrho}}) = \log c + \frac{1-\varrho}{\varrho} \log h + \frac{\zeta}{\varrho} \log M$$

thus

$$u = f_{16}(c^{\varrho} h^{1-\varrho} M^{\zeta}) + A_{11} \log c + A_{12} \log M + F(M)$$

normalize the constants A_{11} and A_{12} to sum to 1 to get:

$$u = \epsilon \log c + (1-\epsilon) \log M + \log v(c^{\varrho} h^{1-\varrho} M^{\zeta}) + F_1(M).$$

Taken together, the two cases imply that the utility function takes the form

$$u(c,h,M) = \begin{cases} \frac{\left(c^{\epsilon} M^{1-\epsilon} v\left(c^{\varrho} h^{1-\varrho} M^{\zeta}\right)\right)^{1-\sigma}}{1-\sigma} + f(M) & \sigma \neq 1 \\ \log\left(c^{\epsilon} M^{1-\epsilon}\right) + \log v(c^{\varrho} h^{1-\varrho} M^{\zeta}) + f(M) & \sigma = 1 \end{cases}.$$

The proof of sufficiency is analogous to the sufficiency proof in the optimal allocation. QED.

D.5 Proof of Proposition 3

Proof. The preferences are separable if the MRS between hours and leisure technologies is independent of consumption. Given the preferences in the required class, we get

$$MRS_{hM} = -\frac{u_h}{u_M} = -M \frac{v'(x)\, x(1-\varrho)}{(1-\epsilon)v(x) + \zeta v'(x) x} = -\frac{M}{h} \frac{\varepsilon_v(1-\varrho)}{(1-\epsilon) + \zeta \varepsilon_v}.$$

where $\varepsilon_v(x)$ is the elasticity of function v with respect to x. Since the MRS is a function of M, h and x only, it is independent of c as long as x is independent of c. This is the case if and only if $\varrho = 0$. □

E The platform pricing decision

The model developed in this Appendix builds on Rochet and Tirole (2003) and Armstrong (2006). The environment is simpler than the problem considered in the main text; it serves to highlight the important issues when it comes to the optimal pricing strategy of platforms operating in two-sided leisure markets. In particular, it shows what kind of considerations may be important in driving low or zero prices. In short, high elasticities of consumer demand and substantial benefits to the other side of the market (advertisers) can lead to the optimal pricing strategy that features zero-price leisure services in equilibrium. These basic insights extend beyond the simple monopoly structure to models of platform competition.

Suppose there are two groups: a unit measure of consumers (group 1) and measure-A of firms / advertisers (group 2), interested in interacting with each other through a monopoly platform. In particular, suppose that the platform provides consumers with leisure technologies of value M and charges price p_1 for accessing the service. Furthermore, consumers may care about how many firms advertise on the platform (with ambiguous sign). The platform charges firms price p_2 for accessing the platform. Since firms use the platform to build brand equity capital, their benefit from using the platform depends on the total time that consumers spend on the platform. Consistent with this description, assume that the utilities of consumers and firms are, respectively:

$$u_1 = \alpha_1 A - p_1 + M + \epsilon \qquad u_2 = \alpha_2 \ell - p_2$$

where $\alpha_2 > 0$, ϵ mean-zero random component, and ℓ is the number / share of consumers that end up using the service. I assume that all agents for whom utility is non-negative participate.

The sign of α_1 is ambiguous as consumers could derive benefits from greater visibility of brands and extra information about their products, but could also find advertising tiresome. To maintain a neutral stance and to make the assumption consistent with the rest of the text, suppose that $\alpha_1 = 0$.

The share of consumers using the platform is a non-decreasing function of utility:

$$\ell = f(u_1) = \phi(\underset{-}{p_1}, \underset{+}{M}).$$

Suppose it costs the platform $\mathbb{C}(M)$ to produce leisure services and brand equity. Furthermore, suppose that there are transaction costs or other frictions that prevent the platform from charging negative prices. The platform then chooses prices and quantity

M to maximize profits:

$$\max_{p_1 \geq 0, p_2 \geq 0, M} \Pi_B = p_1 \ell + p_2 A - \mathbb{C}(M).$$

Given no random component in the utility of the firms, the platform extracts all surplus from the firm side by charging:

$$p_2 = \alpha_2 \ell.$$

We thus have:

$$\Pi_B = p_1 \phi(p_1, M) + A\alpha_2 \phi(p_1, M) - \mathbb{C}(M).$$

Profit maximization implies the following optimality conditions:

$$\phi + p_1 \phi_{p_1} + A\alpha_2 \phi_{p_1} = 0$$

$$p_1 \phi_M + A\alpha_2 \phi_M - \mathbb{C}'(M) = 0.$$

Together with transaction costs, these imply:

$$p_1 = \max \left\{ 0, \frac{\phi + \mathbb{C}'(M)}{\phi_M - \phi_{p_1}} - A\alpha_2 \right\} \tag{205}$$

Equation (205) pins down the optimal price that the platform charges the consumers. Derivatives ϕ_{p_1} and ϕ_M are negative and positive respectively, so the first term on the right hand side is positive. The optimal price is zero if the term $A\alpha_2$ is larger than $\frac{\phi + \mathbb{C}'(M)}{\phi_M - \phi_{p_1}}$. This is more likely when: (i) demand for platform services is low (low ϕ); (ii) it is cheap to produce leisure services (low $\mathbb{C}'(M)$); (iii) consumer demand is highly elastic to prices and leisure technologies (high $\phi_M - \phi_{p_1}$); and (iv) when there are many advertisers whose utility is highly sensitive to the number of consumers using the service (high A and α_2, respectively). Many of these conditions are likely to be satisfied in the context of leisure platforms. This analysis underlies the logic of focusing on free leisure services in the rest of the paper. See Appendix G for how to incorporate paid-for leisure consumption goods into the model.

F Oligopolistic market structure in the leisure sector

An alternative market structure in the leisure sector is an oligopoly with J platforms competing a'la Cournot (J is a parameter). This setup assumes that the implicit features of the two-sided market discussed in Appendix E lead to zero prices in equilibrium.

The main difference to the setup in the main text is that the platforms now face a downward sloping demand curve for brand equity. I continue to assume however that they are sufficiently small so that they do not internalize the impact of their decisions on the aggregates $\bar{b}$ and M. The dynamic problem of platform j is then:

$$\max_{L_{M,j}(t)} \int_0^\infty e^{-\int_0^t r(\tau)d\tau} \left(p_B(t) \cdot M_j(t) \frac{\ell(t)}{M(t)} - w(t) L_{M,j}(t) \right) dt \;\; \text{subject to} \tag{206}$$

$$\dot{M}_j(t) = L_{M,j}(t) A(t)^\phi$$

$$p_B(t) = \alpha^2 \chi \frac{Y}{(A\bar{b})^{\frac{\alpha}{1-\alpha}\chi}} \left(B_j + \sum_{k \neq j} B_k \right)^{\frac{\alpha}{1-\alpha}\chi - 1}$$

$$B_j(t) = M_j(t) \frac{\ell(t)}{M(t)}$$

taking the aggregates M and $\bar{b}$ as given. The current-value Hamiltonian associated with this problem is:

$$\mathcal{H} = p_B(t) \cdot M_j(t) \frac{\ell(t)}{M(t)} - w(t) L_{M,j}(t) + Z(t) \left[L_{M,j}(t) A(t)^\phi \right]$$

where $Z(t)$ is the costate variable associated with the constraint. By the Maximum Principle, the solution satisfies:

$$Z(t) A(t)^\phi = w(t) \tag{207}$$

$$p_B(t) \left(1 - \left(1 - \frac{\alpha}{1-\alpha}\chi \right) \frac{1}{J} \right) \frac{\ell(t)}{M(t)} = r(t) Z(t) - \dot{Z}(t). \tag{208}$$

In equilibrium, demand for brand equity is $p_B(t) = \alpha^2 \chi \frac{Y}{B}$ and brand equity technology is $\ell = B$, so that equation (208) becomes:

$$\alpha^2 \chi \frac{Y(t)}{M(t)} \left(1 - \left(1 - \frac{\alpha}{1-\alpha} \right) \frac{1}{J} \right) = r Z(t) - \dot{Z}(t).$$

Otherwise the structure of the equilibrium is the same as in the main text. The main difference is the presence of the markup $\Psi := \left(1 - \left(1 - \frac{\alpha}{1-\alpha}\chi\right) \frac{1}{J}\right)^{-1}$ that decreases with the number of platforms J. If J is finite, platforms have some market power, charge a gross markup $\Psi > 1$, and make positive profits in equilibrium. Market power diminishes the quantity of brand equity and thus of leisure technologies that are produced. As argued in Section 4, for most parameter values there is under-provision of leisure technologies in equilibrium, and so market power in the leisure sector in that case exacerbates this inefficiency.

G Leisure-consumption complementarities

Consider a more general activity-based framework where each leisure activity requires leisure time $\ell(\iota)$, free leisure services $m(\iota)$ and leisure consumption goods $c(\iota)$. For simplicity, assume that elasticity of substitution between time or leisure services and leisure consumption within activity is equal to one, so that:

$$l := \left(\int_0^M \left[\left(\ell(\iota)^{\varphi} c(\iota)^{1-\varphi} \right) \right]^{\frac{1}{1+\zeta}} d\iota \right)^{1+\zeta}$$

where $\varphi \in (0,1]$. We recover the formulation in the main text by setting $\varphi = 1$. A symmetric allocation of time and consumption across activities implies that

$$l = \left(M \left(\left(\frac{\ell}{M} \right)^{\varphi} \left(\frac{C_L}{M} \right)^{1-\varphi} \right)^{\frac{1}{1+\zeta}} \right)^{1+\zeta} = M^{\zeta} \ell^{\varphi} C_L^{1-\varphi}.$$

To see the consequences of this formulation for labor supply of the household, consider the simple static time allocation problem of a household with labor income only:

$$\max_{C_L, h} \log(wh - p_L C_L) + M^{\zeta} (1-h)^{\varphi} (C_L)^{1-\varphi}$$

The first order conditions are:

$$\frac{1}{C} p_L = (1-\varphi) M^{\zeta} (1-h)^{\varphi} (C_L)^{-\varphi} \tag{209}$$

$$\frac{1}{C} w = \varphi M^{\zeta} (1-h)^{\varphi-1} (C_L)^{1-\varphi} \tag{210}$$

Thus the expenditure shares are constant and:

$$C_L = \frac{1-\varphi}{\varphi} \frac{w}{p_L} (1-h).$$

Plugging this into (210):

$$C = \frac{w}{\varphi M^{\zeta} \left(\frac{1-\varphi}{\varphi} \frac{w}{p_L} \right)^{1-\varphi}}. \tag{211}$$

Combining 211 with the budget constraint we obtain:

$$h = \min\left\{1, 1-\varphi + M^{\zeta}\left(\frac{\varphi}{1-\varphi}\frac{p_L}{w}\right)^{1-\varphi}\right\} \tag{212}$$

$$C = \varphi^{\varphi}(1-\varphi)^{\varphi-1}M^{\zeta}w^{\varphi}p_L^{1-\varphi} \tag{213}$$

$$C_L = \frac{1-\varphi}{\varphi}\frac{w}{p_L}(1-h) \tag{214}$$

Equation (212) shows that the time that households allocate to leisure continues to depend positively on leisure technologies M, so that qualitatively the main mechanisms and hence the implications of the paper go through with that more general formulation. It also shows that in presence of leisure consumption goods, hours worked do not converge to zero but instead to a lower bound of $1-\varphi$. This is intuitive: in the limit, households must afford to buy leisure consumption goods therefore they work more than in the baseline model. Moreover, equations (213) and (214) show that an expansion in leisure technologies acts as a relative demand shifter, boosting demand for consumption goods that are complimentary with leisure and reducing demand for traditional consumption.

H Non-marketable leisure

Suppose leisure output is a combination of marketable and non-marketable leisure, such as hiking or walking in the park. For simplicity, assume that the elasticity of substitution between marketable and non-marketable leisure is one so that:

$$l = l_M^{\eta}\ell_N^{1-\eta}$$

where ℓ_N is time spent hiking, $l_M := \left(\int_0^M \ell(\iota)^{\frac{1}{1+\zeta}}d\iota\right)^{1+\zeta}$ and ℓ_M is total marketable leisure time as before. Since $\frac{\ell_N}{\ell_M} = \frac{1-\eta}{\eta}$ and $l_M = \ell_M M^{\zeta}$ we get

$$l = \ell_M M^{\eta\zeta}\left(\frac{1-\eta}{\eta}\right)^{1-\eta}$$

Labor supply is in this case

$$h = M^{-\eta\zeta}\eta^{-\eta}(1-\eta)^{\eta-1}.$$

Thus all the results of the benchmark framework go through after parameter ζ is recalibrated to reflect the fact that leisure technologies crowd out not just time at work but also time spent on non-marketable leisure.

I Schumpeterian economy with leisure-enhancing technological change

Consider the basic Schumpeterian growth model with constant population of size N. Each household works h hours, with $h = \min\left\{1, \Phi M^{-\varsigma}\right\}$, as in the model in the main text. Final output is given by:

$$Y = \int_0^1 A_i^{1-\alpha} \left(\left(\frac{b_i}{\bar{b}} \right)^{\chi} x_i \right)^{\alpha} di \cdot L_Y^{1-\alpha}$$

where x_i are the intermediate inputs and A_i is the input-specific productivity. Intermediate product demand is:

$$p_i = \alpha (A_i L_Y)^{1-\alpha} \left(\frac{b(i)}{\bar{b}} \right)^{\alpha\chi} x_i^{\alpha-1}.$$

Thus intermediate producer's problem is to

$$\max_{x_i, b_i} \; \alpha (A_i L_Y)^{1-\alpha} \left(\frac{b_i}{\bar{b}} \right)^{\alpha\chi} x_i^{\alpha} - (r+\delta) x_i - p_B b_i$$

which implies equilibrium quantity:

$$x_i = \left(\frac{\alpha^2}{r+\delta} \right)^{\frac{1}{1-\alpha}} A_i L_Y$$

We can thus write the final output as:

$$Y = \left(\frac{\alpha^2}{r+\delta} \right)^{\frac{\alpha}{1-\alpha}} \left(\int_0^1 A_i di \right) L_Y$$

Equilibrium spend on ads is the same as in the main text. Equilibrium profits are therefore:

$$\Pi_i = x_i(p - (r+\delta) - \chi(r+\delta)) = \left(\frac{1}{r+\delta} \right)^{\frac{\alpha}{1-\alpha}} \alpha^{\frac{1+\alpha}{1-\alpha}} A_i L_Y (1 - \alpha - \alpha\chi) = \pi A_i L_Y.$$

Research

Assume that research costs R_i of final output every period. Research is risky. Denote by μ the probability that research succeeds, and by $A^* := \gamma A$ the target productivity level of the successful innovation. Finally, define $n := R/A^*$ as the productivity adjusted expenditure. Then assume that the success function follows:

$$\mu_i = \lambda n_i^\sigma = \lambda \left(\frac{R_i}{A_i^*} \right)^\sigma$$

Note that $\mu_i' = \lambda \sigma n_i^{\sigma - 1}$. Assume for simplicity that a successful innovator operates the technology for one period, and is subsequently removed either by another innovator or, if no innovator succeeds, by a randomly chosen individual. Thus the reward from pursuing research is $\mu_i \Pi_i$ and the entrepreneur maximizes

$$\max_{R_i} \lambda \left(\frac{R_i}{A_i^*} \right)^\sigma \Pi_i - R_i$$

The optimality condition yields:

$$\lambda \sigma n_i^{\sigma-1} \frac{\Pi_i}{A_i^*} = \lambda \sigma n_i^{\sigma-1} \pi L_Y = 1$$

Solving for n_i gives

$$n_i = (\lambda \sigma \pi L_Y)^{\frac{1}{1-\sigma}}$$

and the optimal frequency of success is $\mu_i = \lambda^{\frac{1}{1-\sigma}} (\sigma \pi L_Y)^{\frac{\sigma}{1-\sigma}}$.

Growth

Growth rate of A is computed as follows:

$$A_{t+1} = \mu A_{t,success} + (1-\mu) A_{t,failure} = \mu \gamma A_t + (1-\mu) A_t$$

Thus

$$\gamma_A = \mu(\gamma - 1) = \lambda^{\frac{1}{1-\sigma}} (\sigma \pi L_Y)^{\frac{\sigma}{1-\sigma}} (\gamma - 1).$$

Clearly, there are two channels through which leisure-enhancing technologies affect γ_A. First, since $L_Y = hN$ by labor market clearing, declining hours worked lead to a declining growth rate of traditional TFP through a market-size effect. Second, since π is diminished by $\alpha^2 \chi$ per unit sold, this also lowers the incentives to R&D and thus lowers economic growth. This latter effect is analogous to the level effect working through the lower share of R&D workers in the baseline model.

J Alternative ways of modeling advertising

This Appendix sketches out two alternative ways to incorporate brand equity competition into the monopolistic competition framework. Note first that the final good production function (imposing symmetry in advertising) can be written as

$$Y = \left(\left(\int_0^A x_i^{\frac{\epsilon-1}{\epsilon}} di\right)^{\frac{\epsilon}{\epsilon-1}}\right)^{\alpha} L_Y^{1-\alpha} di$$

where $\epsilon := \frac{1}{1-\alpha}$ is the elasticity of substitution across the intermediate goods. Here I consider two alternatives to the combative advertising assumption in the main text: that advertising shifts the intensity of tastes towards consumption goods (equivalently raises total factor productivity in the final goods sector); and that advertising makes products more differentiated.

Non-combative advertising

Consider first a formulation where advertising is non-combative:

$$Y = \left(\left(\int_0^A (b_i^{\chi} x_i)^{\frac{\epsilon-1}{\epsilon}} di\right)^{\frac{\epsilon}{\epsilon-1}}\right)^{\alpha} L_Y^{1-\alpha} = \int_0^A (b_i^{\chi} x_i)^{\alpha} di L_Y^{1-\alpha}$$

In a symmetric equilibrium with $K = Ax$ and $B = Ab$ we have:

$$Y = (B^{\chi} K)^{\alpha} A^{1-\alpha-\alpha\chi} {L_Y}^{1-\alpha}. \tag{215}$$

This equation shows that this alternative formulation will have important implications for the level and the growth rate of output. Recall that $B = N\ell$, so that output growth will be fastest at low levels of B, when ad spending and leisure hours are growing the fastest. Over time ads cease to be a source of growth; instead, the formulation suggests that output will be growing more slowly (the exponent on A is $1 - \alpha - \alpha\chi$ instead of $1 - \alpha$). The aggregate demand for brand equity in this case is

$$p_B = \alpha^2 \chi \left(\frac{B}{A}\right)^{\alpha\chi-1} \frac{Y}{A}. \tag{216}$$

Thus, there is a multiplier effect: higher brand equity demand raises output and that feeds back into the demand for brand equity. Combining (216) with the free entry condition into the leisure R&D sector and using $B = N\ell$ gives:

$$\frac{wL_M}{Y} = \alpha^2 \chi \left(\frac{\ell N}{A} \right)^{\alpha\chi} . \tag{217}$$

Equation (217) shows that balanced growth is not possible in this case as the cost share of the leisure sector is not constant. This is hardly surprising: in the aggregate, brand equity inherits the non-balanced growth rate of leisure hours; if brand equity generates output directly as in (215), balanced growth in the aggregate is no longer feasible.

Advertising that alters elasticity of substitution across goods

Consider an alternative formulation:

$$Y = \int_0^A (x_i - b_i)^\alpha \, di \cdot L_Y^{1-\alpha}$$

Demand for product i is:

$$x_i = \left(\frac{\alpha}{p_i} \right)^{\frac{1}{1-\alpha}} L_Y + b_i$$

Clearly, advertising shifts demand. But now it also makes demand more inelastic. To see this, note that the elasticity of demand is

$$\left| \frac{\partial x_i}{\partial p_i} \frac{p_i}{x_i} \right| = \frac{1}{1-\alpha} \left(1 - \frac{b_i}{x_i} \right)$$

In this economy brand equity competition exacerbates the monopoly power of firms, raising prices and lowering output, moving the economy further away from the competitive benchmark.

K Alternative calibration of ζ

To illustrate robustness of the main findings to alternative values of ζ, Figure A.8 shows the paths for hours worked and traditional TFP growth for the calibration in the main text, as well as a lower and a higher value of this parameter. While the qualitative conclusions are unchanged, the different calibrations do matter for the quantitative implications.

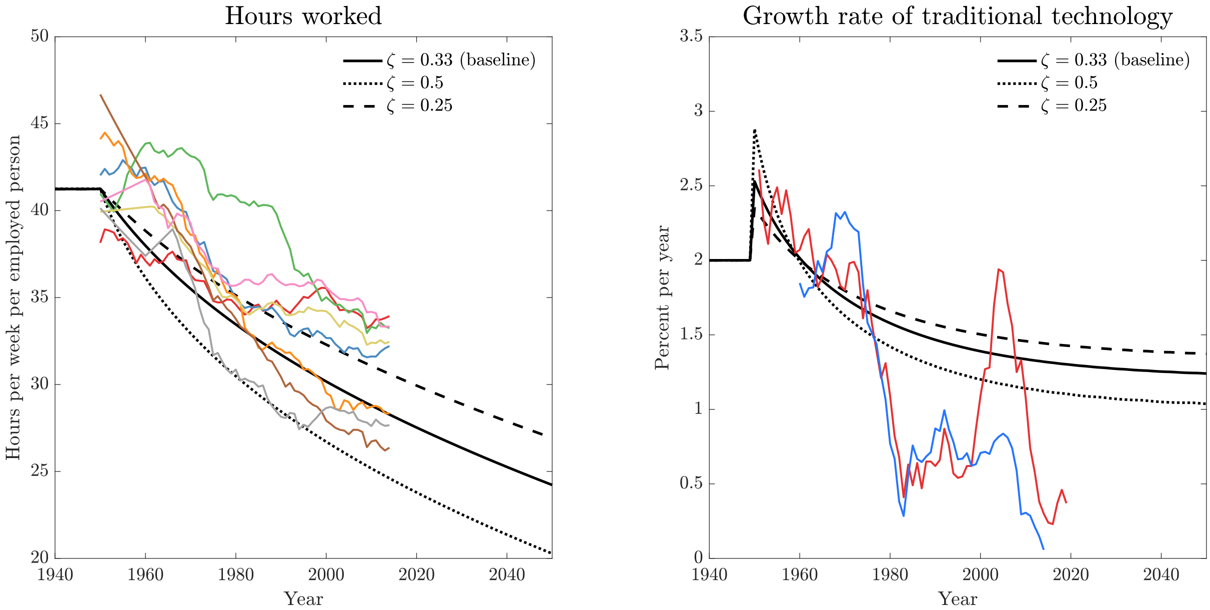

Figure A.8
Robustness to Higher and Lower Values of Elasticity ν.

L Anticipated entry into the leisure R&D sector

The equilibrium concept in Definition 2 incorporates the assumption that the entry to leisure platforms is unanticipated. Figure A.9 presents the solution to the model when the platform entry is instead anticipated. Naturally, segment 1 no longer features exact balanced growth. But these effects are relatively minor, underlying the focus of the main text on the simpler case with unanticipated platform entry.

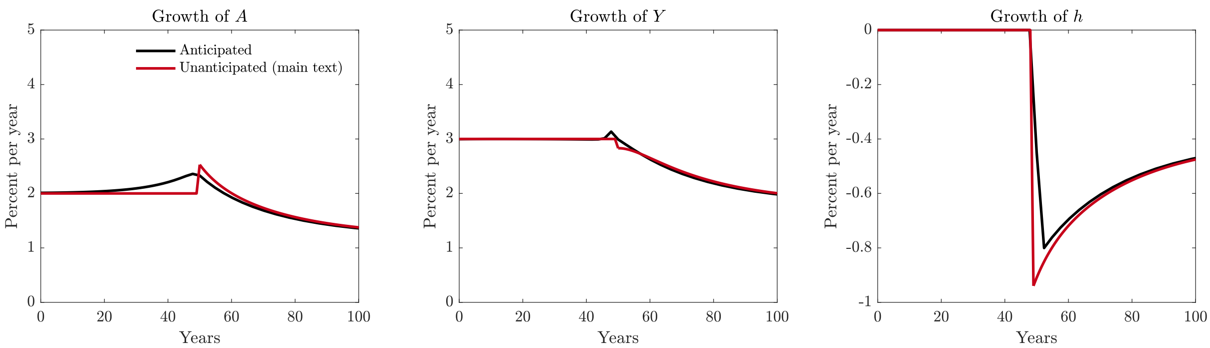

Figure A.9
Transition Dynamics When the Entry of the Platforms is Anticipated